CDS pricing model with CVA – Copula approach

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Abstract

In this article, we propose a general credit derivatives pricing model, based on the work of Cherubini and Luciano (2003) and Luciano (2003). We evaluate a Credit Default Swap with counterparty risk. We include the Credit Valuation Adjustment (CVA), advocated by Basel III in our evaluation to optimize the allocation of economic capital. We work from the general pricing representation established by Sorensen and Bollier (1994) and, unlike in the aforementioned works, loss payment does not necessarily occur at contract expiration. Dependence between counterparty risk and that of the reference entity is approximated by copulas. The CDS's vulnerability in extreme dependence cases is studied, given our decision to use a mixture copula combining common "extreme" copulas. By varying Spearman's rho, the mixture copula covers a broad spectrum of dependence while ensuring closed form prices. The resultant model is adapted to market practices and easy to implement. At the end of this article, we provide an application on real market data.

This article aims to price a single-name or vanilla credit derivative, namely a Credit Default Swap (CDS), including counterparty risk, called a vulnerable CDS. Moreover, we would like to consider in the valuation of CDS spreads the impact of the Credit Valuation Adjustment (CVA), which has become increasingly important in derivatives trading and a part of Basel III prudential requirements imposed by the Basel Committee on Banking Supervision (BCBS). The CVA must reflect the market value of counterparty credit risk (e.g. in the form of a change in the counterparty's credit quality or rating)2. Indeed, a theoretical price that does not factor in the counterparty's vulnerability tends to over- or under-estimate the real premium amounts paid on CDS contracts, leading to an erroneous calculation of regulatory capital needed to avoid undermining a financial institution's solvency.

In view of this new approach to measuring counterparty risk, we are attempting to refine a CDS pricing model proposed by Luciano (2003) which captures the dependence between

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[2] We will return to this topic later.

defaults by copula, a technical tool known for its great quality for measuring dependence between two variables. This tool enables better specification of the dependence structure, expressing any joint distribution function in terms of its marginal distribution functions (Sklar 1959). In the evaluation of credit derivatives, the copula thus enables us to determine the joint default probability in a step-by-step process: model the marginal default probabilities and then specify their dependence. Li (2000) exploited this advantage for the first time via usage of a specific copula: the Gaussian copula. In our model, we have opted for a mixture copula combining linearly common "extreme copulas", more explicitly, the product copula, the minimum copula and the maximum copula. There are a large number of mixture copulas. The one we adopted gives closed form prices in extreme dependence cases; it is defined by a single parameter directly linked to nonparametric dependence measures such as Kendall's tau and Spearman's rho.

The model we are developing, also based on the works of Cherubini and Luciano (2003b), could also be adapted to the valuation of structured products, such as First-to-Default Swaps.

The initial model successfully provided closed-form pricing bounds for the vulnerable CDS and FTD type of credit derivatives to gauge the behavior of premiums with respect to a change in the counterparty's credit quality. We reformulate this problem of spread valuation by using the general CVA equation and by relaxing the assumption made by Luciano (2003) relating to the occurrence of default. While the initial model considers, for the sake of simplicity, that the protection payment in case of default is only possible at expiration of the contract, we offer a more general case whereby default may occur at any time before maturity and protection payment made at the end of each spread payment period. In this way we are trying to adapt the initial spread valuation model to the reality of prevailing financial market practice.

We thus obtain a model that can be easily calibrated on the market. Moreover, by incorporating a stochastic recovery rate, as we plan on doing in the future, it would be entirely adapted to the reforms set out in Basel III. We would also like to point out that this is an easy-to-implement, fairly general model, where default occurrence can be incorporated either via the structural model of credit risk or via a reduced-form one.

This article is structured as follows: in the first section, we will provide a review of literature on credit derivatives pricing using copula-based approaches and on recent works incorporating the CVA in this type of pricing. The second section will focus on the presentation of the CDS premium valuation mode in cases of extreme dependence under the mixture copula. We examine counterparty risk as being the difference between the premium paid on a default-free CDS and that paid on a vulnerable CDS. More explicitly, we will analyze the sensitivity of the premiums with respect to dependence between credit entity and the protection seller also subject to default risk. In the last section, we will provide an application on financial market data collected from Moody's and Bloomberg. We will later put forth a self-criticism and suggest model improvement measures so as to expand its usage, notably, with the incorporation of a stochastic recovery rate.

1. Literature review

Li (2000) was the first to suggest the use of copulas in credit derivatives pricing. He adopted a very specific copula, the Gaussian one, in the context of the structural credit risk model. His idea was later expanded by Schönbucher and Schubert (2001), in the context of the reduced-form model, and where they used copula also to model the dependence between the default triggering thresholds of different obligors by the Gaussian and Archimedean copulas.

Jouanin et al. (2001) further developed the contribution made by Schönbucher and Schubert (2001) and constructed their own approach using the copula to model the dependence between defaults in the context of the intensity model.

Cherubini and Luciano (2002a, 2002b, 2003a, 2003b) offered many applications of the copula in credit derivatives evaluation whereby they adopted mixture copulas to capture the dependence between defaults of different entities and evaluate the counterparty risk in pricing bivariate options, single-name CDS and First-To-Default Swaps. They thus obtained closed-form prices of derivative products and in hedging strategies. They then compare these results with those obtained by the Clayton copula. Other works and articles have also studied the usage of copulas in finance, like Embrechts et al. (2003), Giesecke (2004), Cherubini et al (2004) and a large number of studies to the present day like Cherubini and Romagnoli (2009), Luciano and Semeraro (2010), Cherubini et al. (2011), etc.

While in 2002 and 2003, the above-mentioned works of Cherubini and Luciano shed light on the possibility of counterparty default, it was not known as CVA, the acronym that is in use today. In fact, a credit valuation adjustment (CVA) measures the difference between the present value of a derivative without counterparty default risk and the value when this risk is taken into account.

The general formula can be written:

$$CVA = \sum_{t=1}^{T} \frac{CF_t}{(1+r)^t} - \sum_{t=1}^{T} \frac{CF_t}{(1+r+\Pi)^t}$$

where T denotes the maturity of the contract, CF_t the cash flow expected from the contract at date t, r the risk-free interest rate, and Π the market risk premium (spread) for the counterparty[3].

The higher the counterparty risk (as reflected by a widening of counterparty spreads), the greater the CVA, resulting in a decline in the derivative's market value.

More generally, a credit valuation adjustment, or CVA, measures the difference between the value of a portfolio without counterparty risk and the value of the same portfolio once the probability of counterparty default is taken into account. It means that losses in market value stemming from a higher probability of counterparty default ignored under Basel II will be taken into account under Basel III, especially since it turned out to be massive during the crisis. Indeed, determining the capital required for CVA on the Over the Counter Market (OTC) derivatives is the main innovation of Basel III in the calculation of risk-weighted assets.

We will succinctly present the works that have examined counterparty risk in the evaluation of credit derivatives without going into too much depth on the technical aspects.

We cite Canabarro and Duffie (2003), who propose a practical method for calculating CVA on credit derivatives traded on OTC by considering, in particular, interest rate swaps between two entities which could default.

De Prisco and Rosen (2005) also consider a number of methods, among which, the Monte Carlo simulation and approaches found in Basel I to calculate statistics used to gauge counterparty risk.

Gibson (2005) provides an analytical method based on a Gaussian model for the mark-to-market valuation and a Monte Carlo simulation based on a Gaussian random walk to evaluate the expected exposure and expected positive exposure to the risks of collateral and margin agreements of counterparties.

Redon (2006) also examines a number of methods for calculating expected exposure to factor in Wrong Way Risk (WWR), as named by the Basel Committee. WWR is the risk that occurs when exposure to a counterparty is adversely correlated with the latter's credit quality.

Zhu and Pykhtin (2007) are also interested in gauging expected exposure and CVA in the case of Wrong Way Risk.

[3] Definition of CVA provided by BNP Paribas in "Basel III: no Achilles' spear", Quignon L., *Conjoncture*, May-June 2011

Brigo and Capponi (2008) consider bilateral counterparty risk and propose a general equation for calculating CVA in the absence of joint defaults. They use a trivariate Gaussian copula to represent the risk of the protection buyer on a CDS, that of the protection seller or counterparty and that of the underlying's issuer. They believe that pure contagion models may be inappropriate for modeling CVA since they underestimate risk in a high correlation climate.

Similarly, Brigo and Capponi (2008), Gregory (2009) also seek to measure CVA in a bilateral credit risk portfolio consisting of derivative products traded on the OTC market. With this in mind, Gregory (2009) uses a Gaussian copula model allowing for simultaneous defaults (thereby representing systematic risk). He concludes that the incorporation of simultaneous default risks makes an insignificant contribution to CVA valuation.

Lipton and Sepp (2009) develop an original method for valuating CVA in a CDS contract where credit risk is measured by a structural model based on a jump-diffusion process.

Pykhtin (2011) proposes a general framework for calculating regulatory capital in counterparty credit risk systemically entailing CVA. Banks must henceforth subtract the CVA of the counterparty's Exposure at Default (EAD) value. He uses his model to analyze the treatment of counterparty risk under Basel II and III.

More recent studies continue to consider the measure of CVA and its effects on the result of financial institutions and the capital burden that it will impose on them. An article recently published in Risk Magazine[4] also deals with the impact of the measure of CVA on the capital burden that lending institutions will impose on their sovereign clients for the first time. As these institutions will be required to comply with the new reforms of counterparty risk measures as of 2013, if governments hold back on providing guarantees, the capital costs could turn out to be considerable.

2. Model

2.1. Presentation

Let's look at a simple credit derivative, a CDS. In our model, we limit ourselves to a brief definition of the product to be able to develop its valuation equation.

A CDS is a contract by which a protection seller A receives periodic premium payments from a protection buyer in exchange for which the former commits to compensating the latter in the event of default or other credit event of the reference entity Z, the issuer of the

[4] *In* Carver L., CVA charge will hit sovereign exposures, *Risk Magazine*, July 2011.

underlying, in this case, a bond. Indeed, if there is no default, only the periodic premium is paid in this contract. In case of a credit event, the protection seller will pay the "loss given default (LGD) on the reference bond. In addition to the possibility of reference entity default, we consider our pricing model the possibility of default by either the counterparty or protection seller, making a CDS vulnerable. In such a case, the premium is paid if and only if counterparty A and reference entity Z survive. If they default, only a fraction of the amount due, equal to the counterparty's recovery rate, is paid.

Unlike Cherubini's and Luciano's (2001) assumption of early premium payment, and Luciano's assumption (2003) that payment for loss given default will not occur before contract expiration, we allow for default to occur at any time before contract expiration with protection payment occurring at the end of each premium payment period (regardless of the payment frequency, be they quarterly or other). Moreover, we assume that the spread is paid at the end of the payment period even if the reference entity defaulted during the same period. This situation more closely resembles what actually happens on the market.

Let us notice that the times-to-default of A and Z are defined on a common probability space $(\Omega, \sigma, \mathrm{P})$ with filtration F . The information accumulated in F is related to the solvency of both the insurer and the issuer. With respect to the issuer, we need to know if he has defaulted in the past and, if so, exactly when (default monitoring). As for the insurer, the examination of solvency is just as complex, be it in normal time or in the context of macroeconomic risks. Given that the protection seller is a specialist agent involved in many of the same type of transactions, his financial soundness depends upon: (a) his management competency in normal time (competency in asset/liability adjustments, hedging, portfolio diversification of those insured, etc.); and (b) all the risks of the businesses he covers, therefore, including macroeconomic factors. The better he calculates his exposure, the stronger is his solvency. The interactions between the accumulated insurer and issuer information highlights a number of fairly complex dependence aspects on which our copula function will focus. Since the macroeconomic context weighs heavily and the different dependence aspects can become fairly extensive, in our model, we have limited ourselves to the dependence aspect between the default risks of the issuer and the protection seller.

By Q_i^Z we denote the survival probability of reference entity Z beyond time i and Q_i^A that of counterparty A . The survival probabilities are also assumed to be adapted to the filtration F , as defined below.

As such, the forward value of each payment of the protection leg is

$$\left(\bar{C}\left(Q_{i-1}^{Z},Q_{i}^{A}\right)-\bar{C}\left(Q_{i}^{Z},Q_{i}^{A}\right)\right)LGD_{Z}+RR_{A}LGD_{Z}\left[\tilde{C}\left(Q_{i-1}^{Z},1-Q_{i}^{A}\right)-\tilde{C}\left(Q_{i}^{Z},1-Q_{i}^{A}\right)\right] \tag{1}$$

where LGD_j is the Loss Given Default of entity j is a conditional default probability, assumed constant in our model, and it is equal to $1-RR_j$ where RR_j the recovery rate of entity j is assumed constant. However, notice that the determination of the recovery rate is as complex as determining the protection seller's solvency. It also depends on the insurer's management. In most studies and on the market, it is presented as deterministic or stochastic, but it is a key variable that is very hard to determine.

$\bar{C}(u,v)$ is a survival copula representing joint survival probability, and copulas, $\tilde{C}\left(Q_{i-1}^{Z},1-Q_{i}^{A}\right)$ and $\tilde{C}\left(Q_{i}^{Z},1-Q_{i}^{A}\right)$, are called *flipped copulas* derived from survival copulas, as per

$$\bar{C}\left(Q_{i-1}^{Z},Q_{i}^{A}\right)=Q_{i-1}^{Z}-\tilde{C}\left(Q_{i-1}^{Z},1-Q_{i}^{A}\right) \tag{2}$$

$$\bar{C}\left(Q_{i}^{Z},Q_{i}^{A}\right)=Q_{i}^{Z}-\tilde{C}\left(Q_{i}^{Z},1-Q_{i}^{A}\right) \tag{3}$$

Equation (1), which represents the forward value of each periodic payment of the protection leg, is the sum of the probability that Z defaults multiplied by LGD_Z and the probability that A and Z default multiplied by the recovery rate of A and by LGD_Z .

In equations (2) and (3), we apply the *flipping principle,* as follows: keeping the same notations and based on a general copula, $C(u,v)$, the *flipped copulas* stem from

$$P(X<x)=P(X<x,Y<y)+P(X<x,Y>y)$$

or, in other words, it is equivalent to

$$1-Q_{i}^{Z}=C\left(1-Q_{i}^{Z},1-Q_{i}^{A}\right)+\tilde{C}\left(1-Q_{i}^{Z},Q_{i}^{A}\right)$$

In our case, we start from a survival copula, expressed as

$$P\left(X > x\right) = P\left(X > x, Y < y\right) + P\left(X > x, Y > y\right)$$

and is equivalent to

$$Q_i^Z = \tilde{C}\left(Q_i^Z, 1 - Q_i^A\right) + \overline{C}\left(Q_i^Z, Q_i^A\right)$$

which is, in fact, equation (3). The same reasoning applies to equation (2).

Keep in mind that the usage of copulas as joint survival probabilities in counterparty risk valuation is based on the assumption of no arbitrage, which makes it possible to obtain pricing bounds of the credit derivative (no-arbitrage pricing bounds) and results directly applicable to valuation models in incomplete market, as shown by Cherubini and Luciano (2002a, 2002b, 2003a, 2003b).

Unlike Luciano (2003), we reformulate CDS pricing in terms of survival probability. The reason we use the survival copula, by applying to it the *flipping principle,* is to recover the general pricing representation of a credit derivative with counterparty risk. Indeed, by substituting (2) and (3) in (1), we obtain the general representation, which recalls that of Sorensen and Bollier (1994) for interest rate swap pricing that is used in many recent studies dealing with counterparty risk. Brigo and Mercurio (2006) use it in their general equation for pricing credit derivatives with counterparty risk. This representation, applied to a CDS, is the difference between payment protection in a default-free *CDS* or counterparty risk-free CDS expressed in terms of survival probability and the joint probability of exercise of the CDS and default of the counterparty. The equation (1) is written as follows

$$\begin{aligned}
&\left(Q_{i-1}^Z - Q_i^Z\right) LGD_Z - \left(\tilde{C}\left(Q_{i-1}^Z, 1 - Q_i^A\right) - \tilde{C}\left(Q_i^Z, 1 - Q_i^A\right)\right) LGD_Z \\
&+ RR_A LGD_Z \left[\tilde{C}\left(Q_{i-1}^Z, 1 - Q_i^A\right) - \tilde{C}\left(Q_i^Z, 1 - Q_i^A\right)\right] = \\
&\left(Q_{i-1}^Z - Q_i^Z\right) LGD_Z - \left(1 - RR_A\right)\left(\tilde{C}\left(Q_{i-1}^Z, 1 - Q_i^A\right) - \tilde{C}\left(Q_i^Z, 1 - Q_i^A\right)\right) LGD_Z
\end{aligned} \tag{4}$$

It may also be expressed as

$$\left(Q_{i-1}^Z - Q_i^Z\right) LGD_Z - LGD_A LGD_Z \left(\tilde{C}\left(Q_{i-1}^Z, 1 - Q_i^A\right) - \tilde{C}\left(Q_i^Z, 1 - Q_i^A\right)\right) \tag{5}$$

where

$$\left(Q_{i-1}^{Z}-Q_{i}^{Z}\right)LGD_{Z}$$

is thus the payment protection at the end of each period payment of the spread for a default free CDS while

$$CVA = LGD_A LGD_Z \left(\tilde{C}\left(Q_{i-1}^{Z},1-Q_{i}^{A}\right)-\tilde{C}\left(Q_{i}^{Z},1-Q_{i}^{A}\right)\right) \tag{6}$$

is the Credit Valuation Adjustment. Bear in mind that there is no need or requirement for the two copulas in the difference to be the same, although the difference cannot be non-negative. Probably, there should be a link of the type of *convolution copula,*[5] on which a number of professors are working at the University of Bologna, Italy, including U. Cherubini, F. Gobbi, S. Romagnoli and S. Mulinacci. Presentations of their work on this question were made at Harvard and Yale seminars in late March 2011.

In the foregoing, we were interested in the vulnerable CDS's protection leg. We now seek to calculate the premium payments of the premium leg. Given the notations used above, the forward value of a single payment of the fee leg is

$$W\overline{C}\left(Q_{i}^{Z},Q_{i}^{A}\right)$$

where W is the premium or spread paid on the CDS.
The payments of both legs can be thus discounted, cumulated and equated to recover the value of W that allows for counterparty risk.

[5] Following a discussion of our model with Professor Umberto Cherubini, he told us that the relationship between the two copulas linking the survival probabilities at various times may generate limitations to our pricing model and that this link is currently being studied by the above-mentioned research team. That said, he recommended that we consider, as a first step, that the copulas of the difference in the equation (6) are the same. It is worth noting that this question of the copulas and the survival probabilities at different times essentially depends on the information accumulated in the filtration F .

Given the denoting of B_i as the rate factor, it equals the value at time 0 of a zero coupon bond with maturity i. It is determined by $B_i = \exp(-r_i \times i)$ where r_i is the yield to maturity i of a bond without credit risk. We assume that r_i is independent of default events. Equating the two discounted and cumulative legs until the maturity N of the CDS leads us to:

$$W\sum_{i=1}^{N} B_i \bar{C}\left(Q_{i-1}^{Z}, Q_{i-1}^{A}\right) = \sum_{i=1}^{N} B_i \left(Q_{i-1}^{Z} - Q_i^{Z}\right) LGD_Z - \sum_{i=1}^{N} LGD_A LGD_Z B_i \left(\tilde{C}\left(Q_{i-1}^{Z}, 1-Q_i^{A}\right) - \tilde{C}\left(Q_i^{Z}, 1-Q_i^{A}\right)\right) \tag{7}$$

Consequently, the theoretical CDS spread pricing equation is as follows

$$W = \frac{\sum_{i=1}^{N} B_i \left(Q_{i-1}^{Z} - Q_i^{Z}\right) LGD_Z - \sum_{i=1}^{N} LGD_A LGD_Z B_i \left(\tilde{C}\left(Q_{i-1}^{Z}, 1-Q_i^{A}\right) - \tilde{C}\left(Q_i^{Z}, 1-Q_i^{A}\right)\right)}{\sum_{i=1}^{N} B_i \bar{C}\left(Q_{i-1}^{Z}, Q_{i-1}^{A}\right)} \tag{8}$$

We will now go further into the behavior of the CDS premium with respect to the dependence between the default risks of the guarantor or the counterparty and of the reference entity. We compare the premium behavior of the vulnerable CDS with that of the default free CDS and then specify this behavior by considering a mixture copula in order to obtain pricing equations in cases of extreme dependence.

2.2. Study of premium sensitivity in the case of extreme dependence

Keep in mind that, in cases of extreme value dependence, or more explicitly, cases of perfect positive and perfect negative dependence and in the case of independence, the copula is, respectively, the minimum, maximum and product copula. As shown in many works, like in that of Luciano (2003), in these cases of extreme dependence, even the survival copula coincides with the minimum, maximum or product copula, according to the dependence case.

Now let's consider the case of perfect positive dependence between counterparty and reference entity risk: the survival copula coincides with $C^{+}(u,v) = \min(u,v)$, the *flipped copula*s are equal to:

$$\tilde{C}\left(Q_{i-1}^{Z}, 1-Q_i^{A}\right) = Q_{i-1}^{Z} - \min\left(Q_{i-1}^{Z}, Q_i^{A}\right) = \max(Q_{i-1}^{Z} - Q_i^{A}, 0)$$

$$\tilde{C}\left(Q_i^Z,1-Q_i^A\right)=Q_i^Z-\min\left(Q_i^Z,Q_i^A\right)=\max(Q_i^Z-Q_i^A,0)$$

In this case, the premium, denoted W^+, will be equal to

$$W^+=\frac{\sum_{i=1}^{N}B_i\left(Q_{i-1}^Z-Q_i^Z\right)LGD_Z-\sum_{i=1}^{N}LGD_ALGD_ZB_i\left(\max\left(Q_{i-1}^Z-Q_i^A,0\right)-\max\left(Q_i^Z-Q_i^A,0\right)\right)}{\sum_{i=1}^{N}B_i\min\left(Q_{i-1}^Z,Q_{i-1}^A\right)}$$

Logically, we are led to think that the protection seller, who, we assume, in his role as an insurer, manages his risks for a multitude of insured parties, must have better credit quality. The better he calculates his premiums, the more he is solvent or credit worthy. As such, we assume that he presents default probability lower than that of the reference entity for whom protection is bought. Bear in mind that this holds true, outside of macroeconomic or systemic risk. Consequently, we assume, without loss of generality, that the survival probability of A is higher than that of Z. We thus note that the premium in case of extreme perfect positive dependence depends on neither LGD_A nor Q^A, since it amounts to

$$W^+=\frac{\sum_{i=1}^{N}B_i\left(Q_{i-1}^Z-Q_i^Z\right)LGD_Z}{\sum_{i=1}^{N}B_iQ_{i-1}^Z} \tag{9}$$

In cases of perfect positive dependence between A and Z where their risks are positively correlated and the risk of Z is able to directly impact the solvency of A, it is puzzling but hardly inexplicable that the counterparty has no impact at all on CDS price. Probably the two copulas linking the survival probabilities at different times cannot be considered independently of each other. In fact, this unexpected result might be due to the fact that our model only considers a situation in which both the counterparty and the issuer default in the same period. So our model is missing the case of default of the counterparty and survival of the reference entity. This is very rare, but it still influenced our valuation in case of perfect positive dependence. In this case, the exposure is the substitution cost of the CDS that can be computed as CDSwaption, as in the standard Sorensen and Bollier approach to swap credit risk, which was taken up by Cherubini (2004) for the pricing of swap credit by copulas.

In the case of negative perfect dependence, the survival copula is represented by the lower Fréchet bound copula $C^{-}(u,v)=\max(u+v-1,0)$. We thus have

$\bar{C}\left(Q_{i-1}^{Z},Q_{i-1}^{A}\right)=\max(Q_{i-1}^{Z}+Q_{i-1}^{A}-1,0)$

According to equations (2) and (3), the *flipped copula*s are equal to

$$\tilde{C}\left(Q_{i-1}^{Z},1-Q_{i}^{A}\right)=Q_{i-1}^{Z}-\max(Q_{i-1}^{Z}+Q_{i}^{A}-1,0)$$

$$\tilde{C}\left(Q_{i}^{Z},1-Q_{i}^{A}\right)=Q_{i}^{Z}-\max(Q_{i}^{Z}+Q_{i}^{A}-1,0)$$

Like with Luciano (2003), we figure that, without loss of generality, the sum of the default probabilities of A and Z is less than one. Consequently, the sum of survival probabilities of these two entities must be above one, as expressed by

$$\bar{C}\left(Q_{i-1}^{Z},Q_{i-1}^{A}\right)=Q_{i-1}^{Z}+Q_{i-1}^{A}-1$$

$$\tilde{C}\left(Q_{i-1}^{Z},1-Q_{i}^{A}\right)=1-Q_{i}^{A}$$

$$\tilde{C}\left(Q_{i}^{Z},1-Q_{i}^{A}\right)=1-Q_{i}^{A}$$

Substituting, the premium in case of negative perfect dependence, denoted W^{-}, will be given by

$$W^{-}=\frac{\sum_{i=1}^{N}B_{i}\left(Q_{i-1}^{Z}-Q_{i}^{Z}\right)LGD_{Z}}{\sum_{i=1}^{N}B_{i}\left(Q_{i-1}^{Z}+Q_{i-1}^{A}-1\right)} \tag{10}$$

In this case, the counterparty intervenes in the premium calculation via Q_{i-1}^{A}. It thus has an impact on CDS price. If the counterparty protection seller has high credit worthiness and thus high survival probability, the CDS contract becomes less costly to the protection buyer because the former is unlikely to default at the same time as the reference entity, given the negative dependence between their default risks. This result is consistent with what might seem intuitive, namely that an investor should choose a high credit worthy counterparty to compensate him in case of default of the reference entity in a CDS agreement.

Otherwise, it should be noted that in a context of systemic risks, for example, the sum of default probabilities for A and Z is higher than one, thus, the sum of their survival probabilities is lower than one. In such situation, the premium W^{-} will veer towards infinity.

Under independence between A and Z, the survival copula coincides with product copula $C^{\perp}(u,v)=uv$ thus $\bar{C}\left(Q_{i-1}^{Z},Q_{i-1}^{A}\right)=Q_{i-1}^{Z}Q_{i-1}^{A}$ and the *flipped copula*s are expressed as:

$$\tilde{C}\left(Q_{i-1}^{Z},1-Q_{i}^{A}\right)=Q_{i-1}^{Z}-Q_{i-1}^{Z}Q_{i}^{A}=Q_{i-1}^{Z}(1-Q_{i}^{A})$$

$$\tilde{C}\left(Q_{i}^{Z},1-Q_{i}^{A}\right)=Q_{i}^{Z}-Q_{i}^{Z}Q_{i}^{A}=Q_{i}^{Z}(1-Q_{i}^{A})$$

Consequently, the premium in case of independence denoted $W^{\perp}$ is calculated as follows

$$W^{\perp}=\frac{\sum_{i=1}^{N}B_{i}\left(Q_{i-1}^{Z}-Q_{i}^{Z}\right)LGD_{Z}-\sum_{i=1}^{N}LGD_{A}LGD_{Z}B_{i}\left(1-Q_{i}^{A}\right)\left(Q_{i-1}^{Z}-Q_{i}^{Z}\right)}{\sum_{i=1}^{N}B_{i}Q_{i-1}^{Z}Q_{i-1}^{A}} \tag{11}$$

At this point, the vulnerable CDS premium or fee is worth studying not only *per se* but also in comparison with that of a default free CDS or non vulnerable CDS denoted W_{NV}, all things being equal. We can derive it from equation (8) by considering that the survival probability of counterparty Q^{A} is equal to one, since the latter cannot default. We also use the copula's properties when its arguments veer towards zero or 1, more explicitly, if $C(u,v)$ is a copula, for any u et $v\in[0,1]$,

$$C(0,v)=C(u,0)=0$$

and

$$C(u,1)=u,\ C(1,v)=v$$

thus, having $Q_i^A = Q_{i-1}^A = 1$ and $LGD_A = 0$, CVA is obviously null and the default free CDS premium will be evaluated as follows

$$W_{NV} = \frac{\sum_{i=1}^{N} B_i \left(Q_{i-1}^Z - Q_i^Z \right) LGD_Z}{\sum_{i=1}^{N} B_i Q_{i-1}^Z} \tag{12}$$

We recover the value of the vulnerable CDS premium in case of perfect positive dependence, which turned out to be totally independent of the counterparty's credit worthiness (given the model's limitations as evoked earlier).

In order to compare the two premiums, we study the difference of $W_{NV} - W$ representing the CDS's vulnerability. Luciano (2003) also considers it as the counterparty risk premium.

Again considering the various cases of extreme dependence between the counterparty and reference entity, we obtain a number of results that enables us to appreciate the behavior of $W_{NV} - W$ in each of these cases, using above-calculated values W^+, W^- and $W^\perp$.

In case of perfect positive dependence, if we take into account the Fréchet upper bound for copulas, the difference between the two fees is zero, since the non-vulnerable CDS premium is consistent with that of the vulnerable CDS (as show in the equation (12) and (9). We therefore have:

$$W_{NV} - W^+ = 0 \tag{13}$$

This consequence stems from the inherent limits of the model identified for perfect positive dependence. Luciano's initial model (2003) shows that the difference of $W_{NV} - W$, which has been designated as the counterparty risk premium, is positive.

In cases of negative perfect dependence, we consider the Fréchet lower bound and find the difference of $W_{NV} - W^-$ as

$$W_{NV} - W^- = \frac{\sum_{i=1}^{N} B_i \left(Q_{i-1}^Z - Q_i^Z \right) LGD_Z}{\sum_{i=1}^{N} B_i Q_{i-1}^Z} - \frac{\sum_{i=1}^{N} B_i \left(Q_{i-1}^Z - Q_i^Z \right) LGD_Z}{\sum_{i=1}^{N} B_i \left(Q_{i-1}^Z + Q_i^A - 1 \right)}$$

therefore,

$$W_{NV} - W^{-} = \sum_{i=1}^{N} B_i \left(Q_{i-1}^{Z} - Q_i^{Z} \right) LGD_Z \left(\frac{1}{\sum_{i=1}^{N} B_i Q_{i-1}^{Z}} - \frac{1}{\sum_{i=1}^{N} B_i \left(Q_{i-1}^{Z} + Q_i^{A} - 1 \right)} \right) \quad (14)$$

It is obvious that $Q_i^A - 1 < 0$ then $\sum_{i=1}^{N} B_i \left(Q_{i-1}^{Z} + Q_i^{A} - 1 \right) < \sum_{i=1}^{N} B_i Q_{i-1}^{Z}$, we can thus conclude that the difference of $W_{NV} - W^{-}$ is negative, and consistent with Luciano (2003) results, which are opposed to the naïve intuition, that leads to consider that the non vulnerable fee is always greater than the vulnerable one. In fact, under default of the guarantor, the fee is not paid anymore by the protection buyer. This makes the vulnerable CDS contract more expensive than the non vulnerable one, under negative dependence.

Under independence, we have

$$W_{NV} - W^{\perp} = \frac{\sum_{i=1}^{N} B_i \left(Q_{i-1}^{Z} - Q_i^{Z} \right) LGD_Z}{\sum_{i=1}^{N} B_i Q_{i-1}^{Z}} - \frac{\sum_{i=1}^{N} B_i \left(Q_{i-1}^{Z} - Q_i^{Z} \right) LGD_Z - \sum_{i=1}^{N} LGD_A LGD_Z B_i \left(1 - Q_i^{A} \right) \left(Q_{i-1}^{Z} - Q_i^{Z} \right)}{\sum_{i=1}^{N} B_i Q_{i-1}^{Z} Q_{i-1}^{A}} \quad (15)$$

Its value and sign depend on the interaction between the variables' values of the equation that are related to the counterparty and the reference entity.

2.3. Mixture copula and vulnerable CDS pricing

As you may recall, the mixture copula introduced by Konjin (1959) and discussed at great length by Hurlimann (2003). It has been used in numerous credit risk applications since Li (2000) and, more recently, in Li (2006) where he developed a new CDO pricing approach based on a Gaussian mixture copula model. In fact, the advantage of the closed form prices provided by the mixture copula is that it depends on just one parameter, which is directly linked to non-parametric dependence coefficients, like Kendall's tau and Spearman rho.

The mixture copula is defined in terms of extreme ones, namely the minimum, maximum and product copulas. By linearly combining these copulas, one can obtain the class of mixture copulas. As such, to model a positive dependence, the mixture copula is

$$C(u,v) = (1-\alpha)C^{\perp} + \alpha C^{+} \quad \text{for} \qquad 0 \le \alpha \le 1$$

For negative dependence

$$C(u,v) = (1+\alpha)C^{\perp} - \alpha C^{-} \quad \text{for} \qquad -1 \le \alpha \le 0$$

By varying parameter α, we can explore the whole range of positive dependence (for $\alpha > 0$) and negative dependence (for $\alpha < 0$). This parameter is linked to Kendall's tau by the following relationship

$$\tau = \begin{cases} \alpha(\alpha+2)/3 & 0 \le \alpha \le 1 \\ \alpha(2-\alpha)/3 & -1 \le \alpha \le 0 \end{cases}$$

and to Spearman's rho (that we will adopt in our study) by the following equality: $\rho_S = \alpha$
The mixture copulas have another nice feature: they coincide with their survival copulas, since the extreme copulas (minimum, maximum and product) do. Consequently, since $\bar{C} = C$ for C^{+}, C^{-} and $C^{\perp}$, this is also valid for the mixture copula

$$\bar{C}(u,v) = C(u,v)$$

As we demonstrated in the preceding section, in the case of extreme dependence, we obtained closed form prices for the premium valuation. Indeed, the mixture copula gives us a closed form fee for any dependence level ρ_S.
More precisely, by adopting the mixture copula in case of positive dependence, we have

$$\begin{aligned} \bar{C}\left(Q_{i-1}^{Z}, Q_{i-1}^{A}\right) &= C\left(Q_{i-1}^{Z}, Q_{i-1}^{A}\right) \\ &= Q_{i-1}^{Z}\left[\left(1-\rho_S\right)Q_{i-1}^{A} + \rho_S\right] \end{aligned}$$

And the *flipped copula*s correspond to

$$\begin{aligned} \tilde{C}\left(Q_{i-1}^{Z}, 1-Q_{i}^{A}\right) &= Q_{i-1}^{Z} - \bar{C}\left(Q_{i-1}^{Z}, Q_{i}^{A}\right) \\ &= \left(1-\rho_S\right)Q_{i-1}^{Z}\left(1-Q_{i}^{A}\right) \end{aligned}$$

$$\tilde{C}\left(Q_i^Z, 1-Q_i^A\right) = Q_i^Z - \bar{C}\left(Q_i^Z, Q_i^A\right)$$
$$= \left(1-\rho_S\right) Q_i^Z \left(1-Q_i^A\right)$$

Consequently, the premium in case of positive dependence is determined by

$$W = \frac{\sum_{i=1}^{N} B_i \left(Q_{i-1}^Z - Q_i^Z\right) LGD_Z - \sum_{i=1}^{N} LGD_A LGD_Z B_i \left(1-\rho_S\right)\left(1-Q_i^A\right)\left(Q_{i-1}^Z - Q_i^Z\right)}{\sum_{i=1}^{N} B_i Q_{i-1}^Z \left[\left(1-\rho_S\right) Q_{i-1}^A + \rho_S\right]} \qquad (16)$$

To account for negative dependence between the counterparty and the reference entity with the choice of a mixture copula, thus for $-1 \leq \alpha \leq 0$, we proceed as follows

$$\bar{C}\left(Q_{i-1}^Z, Q_{i-1}^A\right) = C\left(Q_{i-1}^Z, Q_{i-1}^A\right)$$
$$= Q_{i-1}^Z \left[\left(1+\rho_S\right) Q_{i-1}^A - \rho_S\right] + \rho_S \left(1-Q_{i-1}^A\right)$$

and the *flipped copula*s are equal to:

$$\tilde{C}\left(Q_{i-1}^Z, 1-Q_i^A\right) = Q_{i-1}^Z - \bar{C}\left(Q_{i-1}^Z, Q_i^A\right)$$
$$= Q_{i-1}^Z \left[1-\left(1+\rho_S\right) Q_i^A + \rho_S\right] - \rho_S \left(1-Q_i^A\right)$$

$$\tilde{C}\left(Q_i^Z, 1-Q_i^A\right) = Q_i^Z - \bar{C}\left(Q_i^Z, Q_i^A\right)$$
$$= Q_i^Z \left[1-\left(1+\rho_S\right) Q_i^A + \rho_S\right] - \rho_S \left(1-Q_i^A\right)$$

Accordingly, the premium in case of negative dependence is determined by

$$W = \frac{\sum_{i=1}^{N} B_i \left(Q_{i-1}^Z - Q_i^Z\right) LGD_Z - \sum_{i=1}^{N} LGD_A LGD_Z B_i \left[\left(Q_{i-1}^Z - Q_i^Z\right)\left[1-\left(1+\rho_S\right) Q_i^A + \rho_S\right] - \rho_S \left(1-Q_i^A\right)\right]}{\sum_{i=1}^{N} B_i \left[Q_{i-1}^Z \left(\left(1+\rho_S\right) Q_{i-1}^A - \rho_S\right) + \rho_S \left(1-Q_{i-1}^A\right)\right]} \qquad (17)$$

In the next section, we will make use of graphs to display the fee behavior under mixture copula, CVA taken into consideration, as well as its sensitivity with respect to changes in credit quality.

3. Numerical application

In this section, we provide an application of vulnerable CDS pricing, based on Moody's and Bloomberg's financial data, in particular, on credit spreads and recovery rates.

In order to compare our results with those of Luciano (2003), we consider the same parameters and data selected in her model, for the same period. More explicitly, we consider a 5-year CDS contract issued by Z, the reference entity for which the protection is bought and whose senior unsecured debt was rated BBB+ at the time of data collection. The debt of the protection seller A was senior unsecured, like that of Z, and rated A+ by Standard and Poor's. According to Moody's estimates, the recovery rate for the unsecured senior debt was 48.84%. As such, the two entities, A and Z, have a loss given default of 51.16%.

In order to estimate their survival probabilities[6] over a horizon of five years, we consider the spread term structure for their rating class and that of AAA-rated bonds. By definition, the spread is the difference of $r_i^j - r_i$ where r_i^j is the yield to maturity i of entity j, which depends on its rating class, while r_i, is the risk-free rate or that of a bond without credit risk. For the latter, we use, like Luciano (2003), that of AAA class instead of Treasury yields. The article's author bases her choice on the opinion of Duffie and Singleton (2003) who state that choosing a non-Treasury curve has the potential advantage of removing some of the liquidity premium associated with Treasuries over most corporates.

We assumed that A and Z have issued zero coupon bonds B_i^j, $j = A, Z$ of maturity i. These bonds are subject to credit risk. We allow that the recovery rate of entity j, RR_j, is deterministic and constant[7] on the contract period until expiry. The no-arbitrage assumption enables us to establish the following link between risky zero coupon bonds B_i^j and risk-free zero coupon bonds B_i

$$B_i^j = \begin{cases} RR_j B_i & \left(1 - Q_i^j\right) \\ B_i & Q_i^j \end{cases}$$

therefore

[6] While Luciano (2003) models the CDS evaluation as a function of default probability, our reformulation of the pricing problem uses survival probabilities, since we use the general credit derivatives pricing formula which is closer to market practice.

[7] In our study, we continue to consider the recover rate as deterministic and constant. That said, this is one of the major areas of our model we would like to refine in the future. We would approach the dependence between the recovery rates of the entities (and thus their LGD) by the copula.

$$B_i^j = \left(1 - Q_i^j\right) RR_j B_i + Q_i^j B_i$$

where Q_i^j is the (risk neutral) survival probability of entity j with maturity i, given information at time 0, $Q_i^j = \Pr\left(i_j > i \middle| F_0\right)$. For our numerical application, we calculate this probability as follows

$$Q_i^j = \frac{B_i^j - B_i RR_j}{B_i\left(1 - RR_j\right)} \qquad (18)$$

By substituting $B_i = \exp\left(-r_i \times i\right)$ in the equation (18) and the equivalent one for B_i^j, $B_i^j = \exp\left(-r_i^j \times i\right)$, the risk-neutral survival probability Q_i^j may be determined as follows

$$Q_i^j = \frac{\exp\left(-\left(r_i^j - r_i\right) \times i\right) - RR_j}{\left(1 - RR_j\right)} \qquad (19)$$

As a result, we obtain the following risk-neutral marginal survival probabilities

Maturity	A	Z
1	0,9929	0,9907
2	0,9871	0,9774
3	0,978	0,9647
4	0,9704	0,9442
5	0,959	0,9287

Note, however, that these survival probabilities may be simply calculated from the Basel rating transition matrices, which provide both survival probabilities and rating changes on a one-year time horizon.

As for joint survival probabilities, we consider a mixture copula with Spearman's rho ρ_S corresponding to the linear correlation coefficient between the two credits. In our numerical application, we evaluate Spearman's Spearman in terms of its equivalence with the linear correlation coefficient ρ. The latter being equal to 0.5401, $\rho_S = 0{,}5222$ as per $\rho_S = \frac{6}{\Pi} \arcsin \frac{\rho}{2}$. Since ρ_S is positive, the mixture copula is thus equal to $C(u,v) = (1 - 0{,}5222)C^{\perp} + 0{,}5222 C^{+}$. By substituting the variables by their values in the equation (3.16), we obtain a 5-year vulnerable CDS of $W = 0{,}2219$ basis points. The value of the non-vulnerable CDS premium equals $W_{NV} = 0,2222$ basis points, which is slightly higher than that of a vulnerable CDS. Their difference measures the vulnerability of the CDS contract, at 0.003 basis points.

As we discussed earlier, we seek to study by use of graphs the behavior of the vulnerable CDS premium as well as its reaction to changes in credit quality, under the mixture copula and while factoring in the CVA. With this in mind, we allow for a variation in Spearman's rho from -1 to 1 in equations (16) and (17). The results are displayed in graph 1[8]. Indeed, we observe that, under the mixture copula, the CDS premium incorporating the CVA is lower than that of the non-vulnerable CDS in cases of negative and positive dependence, with the vulnerable CDS being much cheaper than in the case of negative dependence. This is consistent with an intuitive view on financial markets.

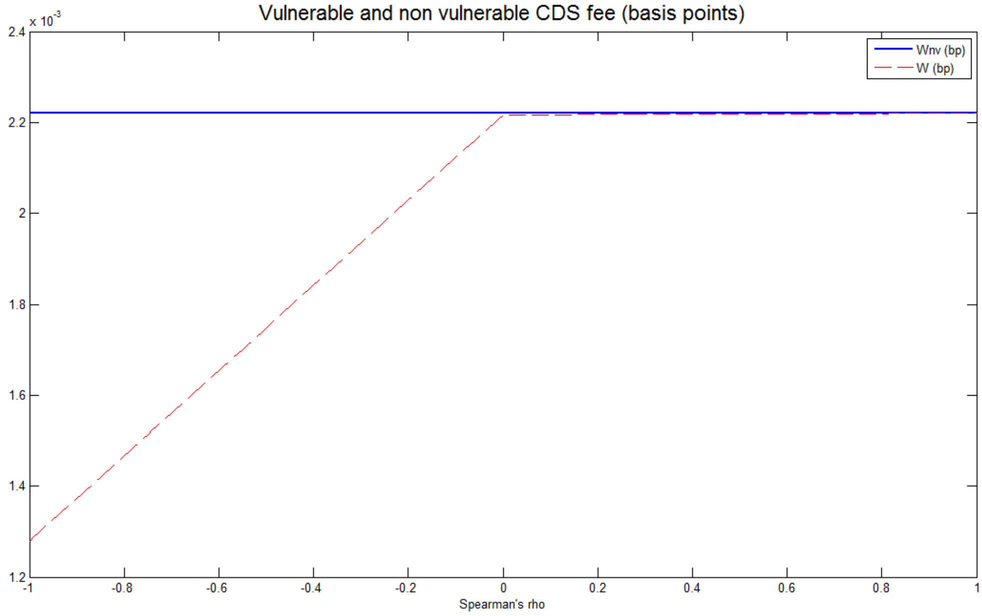
The fact that the premium values in the case of positive dependence appear very close is explained, below, by the limitations of the model. Usually, the more positive the dependence between the two entities, the greater the difference between their two premia. Indeed, although this is not clearly seen in graph 1, the two premia do not cross and the difference widens with ρ_S which veers towards 1. However, as we might expect, the difference is hardly enormous.

Our results are consistent with those obtained by Luciano (2003) for values ρ_S from (-0,4) where the vulnerable CDS premium remain lower than that of non-vulnerable CDS $W < W_{NV}$ but in cases of positive dependence, the vulnerable CDS premium declines more steeply than that of our model.

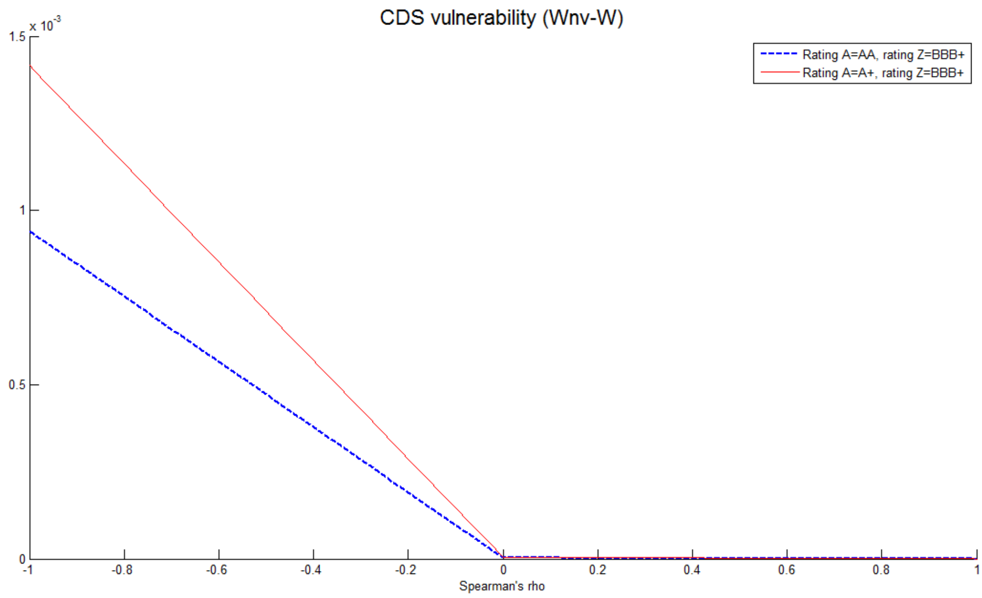
We also study the behavior of $W - W_{NV}$ which measures the vulnerability of the CDS contract or the impact of the CVA's incorporation in the spread calculation with respect to a change in the protection seller's rating. We compare our original case in which counterparty A has an A+ rating and the case where A has a better rating (e.g. AA). We used the AA-AAA spreads on Moody's for said period, and we calculated the risk-neutral marginal survival probabilities in order to compute the vulnerable CDS premium after accounting for the rating change. Graph 2 display the behavior of $W - W_{NV}$, given the rating change.

As with Luciano (2003), the difference between the two premia with a higher AA rating is less than that calculated for the protection seller's A+ rating, regardless of the rank correlation. This remain intuitive since the CDS becomes more expensive when the insurer is more credit worthy, making the difference of $W - W_{NV}$ for AA less than that for A+.

[8] The model was calibrated on Matlab 7.8.0 (R2009a).



Graph 1: Vulnerable CDS premia (W) and that of non-vulnerable CDS (Wnv)



Graph 2: CDS vulnerability – change in insurer's rating

4. Conclusion

From the theoretical point of view, we have shown that a vulnerable CDS's premium may be higher than that of a default free CDS in the case of negative perfect dependence (in opposition to general intuitive view on financial markets), lower in the case of perfect positive dependence[9] and may be lower or higher in the case of independence. The choice of the mixture copula gives us closed form prices in the premium evaluation.

From a practical point of view, our numerical application shows us that the vulnerable CDS premium is generally lower than that of a default free CDS, regardless of the rank correlation. The factoring in of the CVA thus changes downward CDS prices.

To further enhance the contribution of this CDS pricing model, in the future, we would like to refine it by studying the link between the survival probabilities copulas at different times, given our realization that considering them independently in our model influenced our CDS valuation in case of perfect positive dependence. We would also factor into our pricing model for the (rare) case in which the protection seller defaults and the reference entity survives. However, our main contribution would be to include the relevance of stochastic recovery rates in the framework of Basel III pricing and the regulation with a special focus on the CVA. This would make our pricing model closer to market practice. In the latter case, it would be interesting to investigate and address the dependence structure of the recovery rates and thus the LGD figures as well as the dependence between LGD and the survival probability, particularly for the reference entity.

[9] Given the limitations encountered in this case of dependence, we have obtained equality between the two premia, thus zero difference in $W - W_{NV}$.

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