A model of viability for a monetary economy

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1 Introduction

A pure monetary exchange economy is populated by individuals who are free to choose their specialization of production, to buy the corresponding inputs and the goods they consume or accumulate. Nobody can force anyone to buy or to sell: relations between individuals are prompted by voluntary exchange which is another way to that equivalence is the rule. Exchanges do not take place but through money. Individuals get their means of payments from an institution (Mint, banking system, etc.) by monetization of part of their wealth (coinage of precious metals in metallic systems or against promises of future payments in case of credit, etc.)[1].

Being decentralized, the decisions taken by individuals are generally not spontaneously mutually compatible. Regulatory mechanisms are necessary to remedy these incompatibilities. The so-called 'law of supply and demand' is most often invoked. This will be different in this paper where the monetary character of the economy is exclusively considered. Individuals do not barter but buy or sell, spend or receive diverse amounts of means of payment. The principle of equivalence requires that no individual buys more than he sells. But, even if transactions desired by any individual comply *ex ante* with the rule of equivalence, effective transactions make appear monetary imbalances *ex post* which are called disequilibria of balances of payment in the case of international transactions. In any case these balances have to be settled. As a consequence of decentralization of individual actions, equivalence is imposed only *ex post.* An individual (or group of individuals) who has spent more than he has received is constrained to settle the deficit by giving an equivalent part of his wealth or by borrowing an equivalent amount from individuals in excess (either by direct finance or through intermediaties and financial markets).

The settlement constraint is generally considered as providing a stabilizing mechanism which is to money flows what the law of 'law of supply and demand' is to commodity markets. The basic idea runs as follows: a deficit individual

[1] Wage-earners are not considered as individuals since they do not decide what or how to produce. Therefore by individuals we mean independent producers or entrepreneurs or firms, etc. Wage-earners take part into production under the control of entrepreneurs against wages they freely spend in the market. They are not responsable for what entrepreneurs do. Their relation to the market is indirect and conditioned by what entrepreneurs decide.

constrained to become poorer or more indebted is led to decrease his expenditures, which is supposed to lessen his future deficit whereas an excess individual who is better off will tend to spend more, which is supposed to decrease his future excess balance. The famous *gold specie mechanism* may be interpreted along this line when associate with the quantitative theory of money.

For three centuries at least the fundamental problem of a market economy is the capability of the 'anonymous forces of the market' ('law of supply and demand' or mechanism of the balances settlement) to ensure an efficient regulation. Ricardo and Say, although they did support opposed value theories, shared the same optimistic view about market stability opposed to that of Malthus and Sismondi, and later Marx. Until recently, orthodox and heterodox economists fight against each other on the same ground. But, since the 1970's, mainstream economists have ceased to care explicitly about the study of global stability as a consequence of the negative results of the general competitive equilibrium theory in that field. Heterodox economists have deserted the domain of pure theory leaving a deserted field to absent orthodoxes.

The purpose of the present paper is to suggest not only that it is of utmost importance to keep working on global stability and on market economies ability to regulate themselves but also that it is worth exploring an alternative approach since the traditional one has failed.

2 The model

2.1 Generalities

Let be a market economy where H individuals (or decision centers), indexed by $i = 1, ..., h, ..., H$, hold at (t) an amount of wealth (expressed in monetary unit) $\varpi_h(t)$.

Usually wealth is a vector of quantities of the different commodities. Wealth is measured by the scalar product of prices by quantities. Here wealth is defined as the quantity of a special commodity, let call it *minting basis*, which allows to get means of payment from a special institution (Mint or banking system). For instance, in a pure metallic monetary sytem, gold is the unique wealth. In this case, instead of a minting basis, it is possible to speak of *legal money*, that is a money which put and end to transactions. When a dematerialized legal money circulates (say banknotes of the European Central Bank), wealth and legal money are synonymous.

But individuals may get private means of payment from banks at (t). They have to pay back these means of payment at the end of the market $(t + \rho)$, out of their sales.

Wealth allows his holder to get a determinate amount of means of payment at (t) and to finance his expenditures in the market during a period (ρ). The maximum degree of monetization of the wealth is $\varphi_h(t)$. Variable $\varphi_h(t)$ indicates the importance of private or credit money. The $\varphi_h(t)$'s are supposed to be fixed by a monetary authority.

The amount of means of payment an individual may get by monetization of his wealth is the simplest measure of his capacity to intervene in the market. A special case is $\varphi_h(t) = 0$ for all h and all (t). It may be interpreted as describing a strict Gold Standard where gold is the unique wealth. When $\varphi_h(t) > 1$ individuals can spend more than their own wealth. This is the case in a general credit economy.

Let consider a simple monetary economy where individuals voluntarily make payments to each other. A voluntary payment from h to k is denoted by m_{hk}. The receipts of any individual come from other individuals expenditures $\sum_k m_{kh}$ whereas the total expenditures of any individual is $\sum_k m_{hk}$. We have obviously: $\sum_h \sum_k m_{kh} = \sum_h \sum_k m_{hk}$.

Payments matrix of period (ρ) is:

$$M(\rho) = \begin{pmatrix} 0 & m_{12}(\rho) & \cdots & m_{1H}(\rho) \\ m_{21}(\rho) & 0 & \cdots & m_{2H}(\rho) \\ \cdots & \cdots & \cdots & \cdots \\ m_{H1}(\rho) & m_{H2}(\rho) & \cdots & 0 \end{pmatrix}$$

At the end of each period (ρ), each individual has to comply with equivalence principle: his total payments $(\sum_k m_{hk})$ must be equal to total receipts $(\sum_k m_{kh})$. When *voluntary* payments do not exhibit that property, *forced* payments restore equivalence. Concrete forms of forced payments depend on the type of monetary organization.

Consider a market economy where gold is the unique socially recognized wealth. Gold endowments are the exclusive means to get legal means of payment from the Mint (to keep the story simple, monetization is supposed to be costless and free from any seignoriage). In this case gold has a legal price expressed in the monetary unit, say the euro. Circulation of gold coins is the unique possibility for transfering euros from an individual to another. In this system, quantities of gold and euros are transferred as the same time even if gold and money (euros) are not to be confused..

During period (ρ), individual h cannot spend more than the amount of euros corresponding to the coinage of his gold endowment. If an euro is defined by a weight of αgr gold (which means that the legal price of gold is $1/\alpha$ per gr), individual h cannot spend more than $g_h/\alpha = \varpi_h/\alpha$ euros which is the value in euros of his gold endowment: $\sum_k m_{hk} \leq g_h/\alpha$. Let suppose that h's voluntary expenditures are greater than his receipts. What will happen? In such a system the answer is: nothing! The negative balance in euros of individual h has been settled by the transfer of gold contained in the coins that h has paid in excess over his receipts. During the period voluntary payments have redistributed gold endowments. Excess individuals have gained some quantities of gold and deficit agents (h in our example) have lost what the others have gained. In a pure metallic system voluntary payments (in euros) and forced payments (in gold) take place at the same time. Monetary balances are *ipso facto* settled by the gold (monetizable wealth) contained into the coins (legal means of payment). Conceptually (not necessarily concretely true), coinage of gold (creation of money) opens circulation and melting the coins (cancellation of money) closes it.

A remarkable feature of any pure metallic system (without seignorage) is that no individual can run into bankruptcy. Even if an individual has no receipts, it cannot happen that his expenditures exceed his wealth. Whatever may be the dynamics of a market economy under a pure metallic system, the viability constraint $g_h(t) \geq 0 \; \forall h, (t)$ is never violated. Such property is certainly responsible for the fascination that Gold standard has had and still has on some people.

Although very special, pure metallic systems exhibit a fundamental feature common to all monetary systems:

Proposition 1 *Transfers of the mintage basis (here gold) settle interindividual balances; the mintage basis is what society recognizes as wealth expressed in monetary units (euro); this proposition holds valid when capital instead of gold is the mintage basis .*

Consider now the same economy with the following unique modification: instead of legal coins, individuals may finance their transactions using *promises to pay* with legal coins at the end of the period. We will check that proposition 1 holds true when the monetary system is based on credit. In such a system, individual h gets his private means of payment by borrowing from a bank. Take the simplest operation: h borrows at (t), starting point of period (ρ), an amount $m(\rho)$ for financing part or all his voluntary operations during the period. The bank agrees to lend that sum if and only if h credibly promises to pay back a sum greater than $m(\rho)$, say $m(1 + r\rho)$ at $(t + \rho)$. This means that at (t) the bank recognizes that h holds some wealth and accepts to monetize it. That precise wealth is nothing but the present value of $m(1 + r\rho)$, which is $m(1+r\rho)/(1+r\rho) = m(t)$. The wealth the bank accepts to monetize is what is usually called *capital*. Instead of coining a piece of gold, the bank 'coins' a piece of capital. Of course, capital differs from gold in that capital is not tangible. It results from a private agreement. But that agreement is part of an institutional arrangement where a monetary authority (central bank) plays a role. In both cases, means of payment are issued in a supra-individual framework. Individuals experiencing deficits are forced to borrow from excess individuals if they will not go into bankruptcy. That forced operation clearly means a capital loss for deficit individuals and a capital gain for the others (measured by the present value of the future flows of repayment.

Gold and capital are two possible minting basis allowing an issuance of means of payments (coinage or credit) and their cancellation (melting of coins or reimbursement of credit). Proposition 1 appears to be very general[2].

[2] Land is another example of a possible minting basis. In 18th century many propositions for reforming the monetary systems were based on land banks.

2.2 A mixed monetary system

Suppose a mixed monetary system where credit allows individuals to get rid of gold endowments constraint. Let $m_h(t)+\varphi_h(t)$ be the quantity at (t) for market (ρ) of means of payment that h may use for financing his desired transactions (legal coins *plus* promises to pay with legal coins at the end of the market). Now the payment constraint is $\sum_k m_{hk} \leq m_h+\varphi_h$ (instead of $\sum_k m_{hk} \leq m_h$ in a pure metallic system). Suppose for the sake of simplicity that all individuals use all the means of payment they get either toward other individuals or to thelmselves (hoarding). $\sum_k m_{hk} = m_h + \varphi_h$. It is straightforward that if h gets no receipts, his default payment is φ_h. More generally, his receipts must be greater than φ_h to avoid going into bankruptcy. The property of being bankruptcy proof, specific of pure metallic systems, no longer holds as soon as credit is effective.

The payment matrix is now:

$$M(\rho) = \begin{pmatrix} (m_{11}+\varphi_{11})(\rho) & (m_{21}+\varphi_{21})(\rho) & \cdots & (m_{H1}+\varphi_{H1})(\rho) \\ (m_{12}+\varphi_{12})(\rho) & (m_{22}+\varphi_{22})(\rho) & \cdots & (m_{H2}+\varphi_{H2})(\rho) \\ \cdots & \cdots & \cdots & \cdots \\ (m_{1H}+\varphi_{1H})(\rho) & (m_{2H}+\varphi_{2H})(\rho) & \cdots & (m_{HH}+\varphi_{HH})(\rho) \end{pmatrix}$$

Payment matrix could be expressed also in the *minting basis* as $\alpha(\rho)M(\rho)$ to make precise the relation between current transactions and the redistribution of gold.

Let suppose that proportions of payments from h to k during period (ρ) are continuous and derivable functions of money endowments $0 \leq f_{hk}(m_h(t)+\varphi_h(t)) \leq 1$. Current transactions of period (ρ) modify h's wealth as follows[3]:

$$\frac{m_h(t+\rho)-m_h(t)}{\rho} = \underbrace{\sum_{kh} f_{kh}(\cdot)\max(0,(m_k(t)+\varphi_k(t))}_{\text{receipts of } h} - \underbrace{\max(0,(m_h(t)+\varphi_h(t)))}_{\text{expenditures of } h}$$

Making ρ tend to zero and defining $[(m_k(t)+\varphi_k(t))]^+ := \max(0,(m_k(t)+\varphi_k(t))$ allows to rewrite the system giving the evolution of legal money endowments as:

$$\begin{pmatrix} m_1'(t) \\ \cdots \\ m_H'(t) \end{pmatrix} = \begin{pmatrix} f_{11}[(m_1(t)+\varphi_1(t))]^+ - 1 & \cdots & f_{H1}[(m_1(t)+\varphi_1(t))]^+ \\ \cdots & \cdots & \cdots \\ f_{1H}[(m_1(t)+\varphi_1(t))]^+ & \cdots & f_{HH}[(m_H(t)+\varphi_H(t))]^+ - 1 \end{pmatrix} \begin{pmatrix} [(m_1(t)+\varphi_1(t))]^+ \\ \cdots \\ [(m_H(t)+\varphi_H(t))]^+ \end{pmatrix} \tag{1}$$

with $\sum_h m_h(t) = Cte = 1$, and $\sum_h m_h'(t) = 0$ for all (t).

Let $F(\cdot)$ be the matrix of circulation coefficients, $m(t)$, $m'(t)$ and $\varphi(t)$ being the vector of $m_h(t)$, $m_h'(t)$ and $\varphi_h(t)$ respectively. System (1) may be written as:

$$m'(t) = F[(m(t)+\varphi(t))]^+ - I)[(m(t)+\varphi(t))]^+ \tag{2}$$

[3] We remind the reader that $\sum_h (1+\varphi_h) f_{hk}(m_h(t))m_h(t) = (1+\varphi_h)m_h(t)$ where $(1+\varphi_h)f_{hh}(m_h(t))m_h(t)$ is h's hoarding (means of payment non spent).

System (2) has at least one stationary solution $m*$. In the traditional approach the question is to determine global stability properties of that simple monetary economy, for constant φ_h. The intuition is that system (2) is self-regulated since any deficit individual looses and any excess individual gains some purchasing power (φ_h's are constant). According to that story, market punishes individuals who have spent too much and rewards the others allowing them to spend more in the future. That simple idea, very fashionable in the present time of financial crisis, is formally mor or less the same as that of the so-called 'law of supply and demand'. We know that the latter has not the merits that the Vulgate of economists would make us to believe. Some thereticians have demonstrated at the beginning of the 1970's that Walrasian tâtonnement is not generally globally stable in the Arrow-Debreu's model. For analogous reasons, the spontaneous regulation of system (2) is less general than economists would have desired. The conditions under which that system is globally stable have no reason to be met in general.

2.2.1 Viability in a nutshell

Therefore it makes sense to abandon the traditional approach in terms of asymptotic global stability and to explore an alternative way less connected with a social liberal philosophy but more relevant, that of *viability*. Instead of researching the conditions under which system (2) converges toward a stationary equilibrium, it seems more sensible to determine a domain of viability, that is a set of situations of tolerable disequilibria. By tolerable disequilibria we mean any situation in which some fundamental constraints are not violated. For the sake of simplicity we admit that viability means a situation in which no individual goes bankrupt, that is:

$$\begin{array}{c} m_h(t) \geq z \ \forall h, (t) \\ \sum_h m_h(t) = 1 \end{array} \tag{3}$$

where z is a metavariable (see below).

The constraint set K is defined by

$$K := \left\{ (m, z) \in \mathbb{R}^H \times \mathbb{R} \mid m_h - z \geq \ 0, \sum_h m_h = 1 \right\} \tag{4}$$

A viable situation is any $m(t) \in K$.

Let consider a subset of K. If from each point of that subset starts at least at (0) a trajectory $m(t)$ such that $m(t) \in K$ for any (t), we will say that the subset is a viability set. The greater viability set is the viability kernel $Viab(0)(K(0)$. It does not means that the system is globally stable (a trajectory may converge toward equilibrium and be non-viable if it violates at some (t) the viability constraint (3)) but only that starting from $Viab(0)(K(0)$ it exists a manipulation of the controls $\varphi_h(t)$ keeping the economy in the constraint set for any $(t) > 0$.

The size of the viability kernel is a measure of the instability of the economy. Unfortunately, there is no general analytical solution for determining the viability kernel. Numerical simulations are the only method to evaluate the size and the form of the viability kernel[4]. Before embarking upon that task it is important to remind the reader of an important property of the model.

2.3 A fundamental property

A fundamental property is the following :

Proposition 2 *If $\forall h$, $\varphi_h \equiv 0$, then from any initial position $m(0)$in $K(0)$, the solution $m(\cdot)$ remains in $K(0)$. In other words,*

$$Viab(0)K(0) = K(0)$$

Whatever may be the dynamics of a market economy under a pure legal money system, the viability constraint is never violated. Such property is certainly responsible for the fascination that Gold standard has had and still has on some people. In terms of the model, the viability kernel is the simplex

Starting from that proposition it seems reasonable to expect that the size of the viability kernel will reduced as soon as a sufficient amount of credit money is aded to legal money. As we shall see, this intuition needs some qualifications. Another intuition is that the size of the viability kernel depends negatively on the degree of harshness of the monetary constraint that is on z. In order to explore these questions, we have first to develop a numerical example.

3 A numerical example

Let consider the following matrix of coefficients circulation:

$$F(\cdot) = \begin{pmatrix} -1 & 0.3(m_2+\varphi_2) & 0.6(m_3+\varphi_3) \\ 0.3(m_1+\varphi_1) & -1 & 1-0.6(0, m_3+\varphi_3) \\ 1-0.3(m_1+\varphi_1) & 1-0.3(m_2+\varphi_2) & -1 \end{pmatrix}$$

In such an economy the relative importance of individuals seems to crucially depend on the global importance of credit: when the φ_h's are important individual 1 benefits from most part of the expenditures of individuals 2 and 3, individual 2 receiving most part of individual 1's expenditures. It is the reverse when the φ_h's are closed to zero. Individual 3 is then in a good situation and individual gets an important fraction of individual 3's expenditures.

[4]For a complete view of that theory, see [1]

The model is now:

$$\begin{pmatrix} m_1'(t) \\ m_2'(t) \\ m_H'(t) \end{pmatrix} = \begin{pmatrix} -1 & [0.3(m_2+\varphi_2)]^+ & [0.6(m_3+\varphi_3)]^+ \\ [0.3(m_1+\varphi_1)]^+ & -1 & [1-0.6(0,m_3+\varphi_3)]^+ \\ [1-0.3(m_1+\varphi_1)]^+ & [1-0.3(m_2+\varphi_2)]^+ & -1 \end{pmatrix} \begin{pmatrix} [(m_1(t)+\varphi_1(t))] \\ [(m_2(t)+\varphi_2(t))] \\ [(m_H(t)+\varphi_H(t))] \end{pmatrix}$$

$$z'(t) = \gamma(z) \tag{5}$$

where $\gamma(\cdot)$ will be chosen depending what measurement we aim at.
The state constraints are given by (4).
The control constraints are:

$$U = \left\{ \varphi \in \mathbb{R}^H \mid \varphi_h \geq 0, \sum \varphi_h \leq c, \text{ and } \varphi_h = 0 \text{ if } m_h < 0 \right\} \tag{6}$$

3.1 Viability and hardness of the monetary constraint

When it is about balances of payments, be that of an individual r of a country, a crucial point is the hardness of the monetary constraint imposed to the agents. It is possible to imagine for instance that the monetary authority accepts to cover any negative balance upon a certain amount. In terms of the definition of the constraint set z denotes the hardness of the constaint.

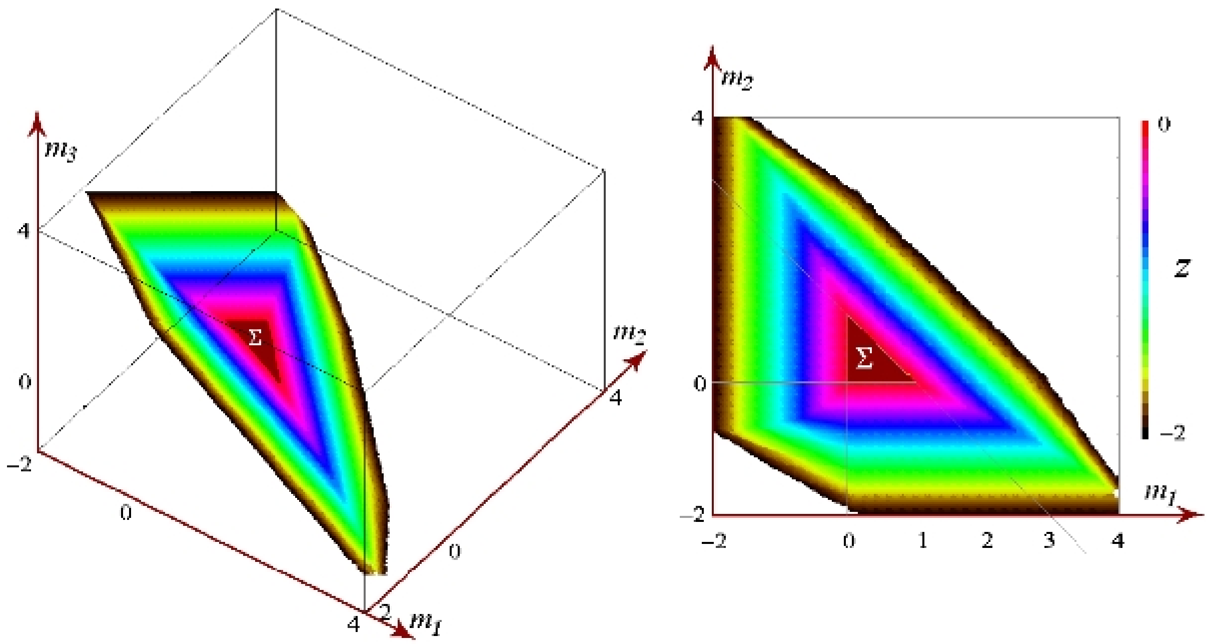
Choosing $\gamma(z) = -z$, $z \in [-2, 0]$, which means that an individual with a quantity of legal money greater than -2 is considered as viable. Taking successive values for z and comparing the size of the viaility kernel for these different values generates viability kernels with a decreasing size as shown in figure below. When $z = 0$, the viability kernel is the simplex in accordance with our proposition above.

In order to get idea of the dynamics we show below a trajectory of our economy. At ($t = 0$) individual 2 is in a bad position and would have ben eliminated (and the economy whould have violated the viability constraint), the monetary constraint had been more severe (-1 for instance). But, thanks to the soft constraint, the economy eventually gets in a zone where all individuals have a positive wealth. The evolution of the controls is shown as well:

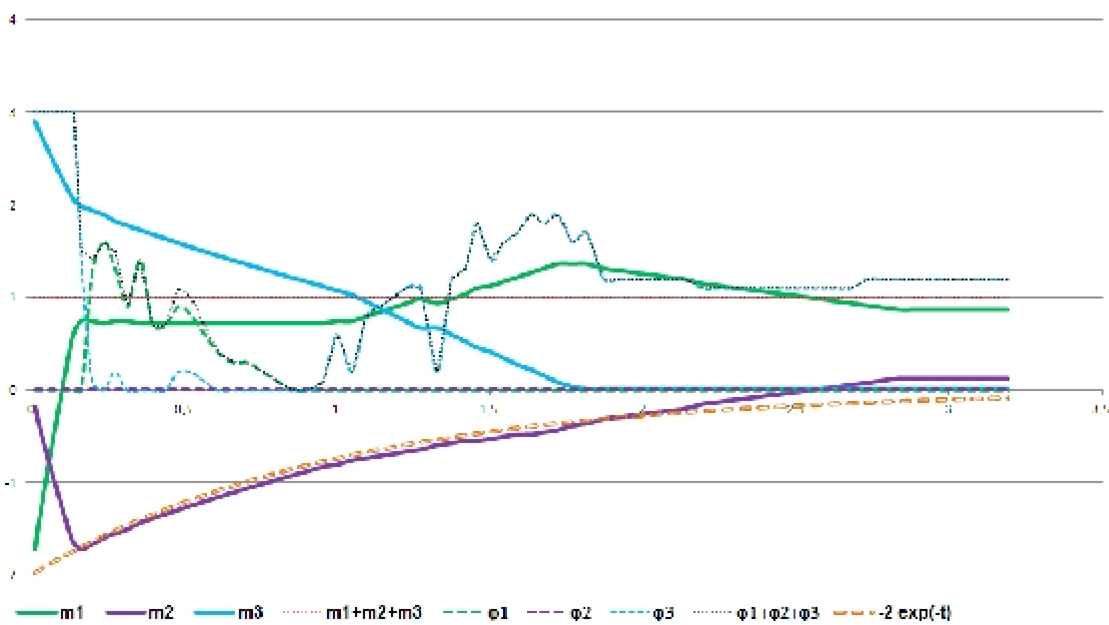
An anologous result would be obtained if we had allowed for a progressive hardening of the constraint as figure belows makes it clear:

3.2 Viability and credit

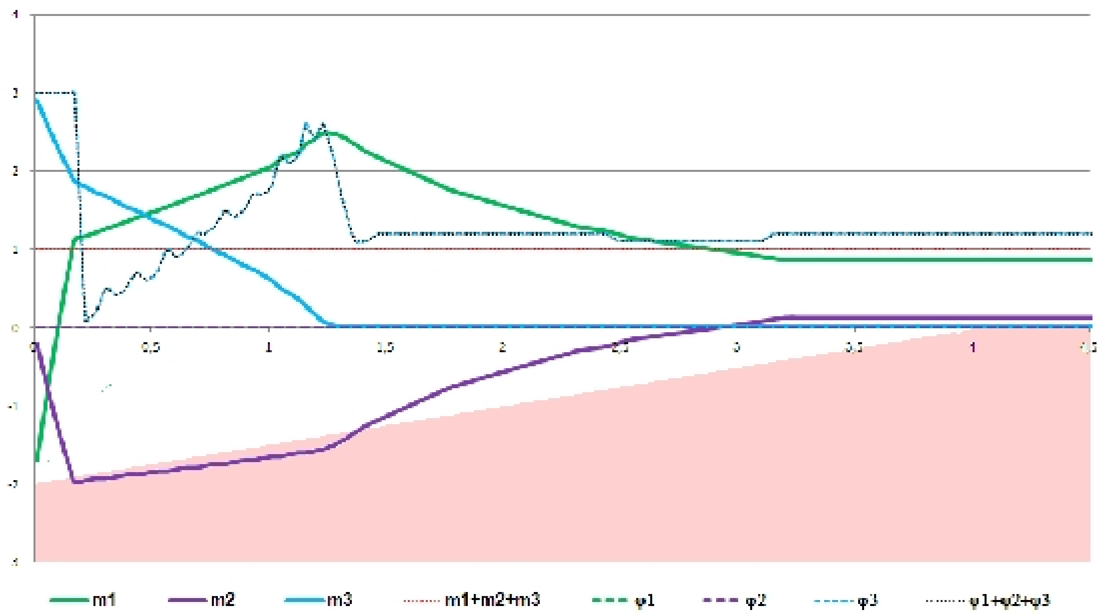
We have recalled above that a monetary where only legal money circulates, which means that nobody can spend more than his legal money endowment, would never experience viability violations. This proposition could make believe that if we introduce more and more credit money as a whole, the viabilty kernel should be reduced progrssively. However we should not forget that viability,



m_3
4
0
−2
0
4
m_1
m_2
4
0
Σ
m_2
4
0
−2
−2
0
1
2
3
4
m_1
0
z
−2



m1
m2
m3
m1+m2+m3
φ1
φ2
φ3
φ1+φ2+φ3
-2 exp(-t)



which is in some sense a generalization of equilibrium, depends on relative wealth of individuals and not on absolute amounts of their spending. In other terms, if the constraint on the controls is not too strict, there are likely control values which ensure vability whatever the global quantity of credit C may be.

Our numerical simulations confirm this second intuition. The computation of the viability kernel gives:

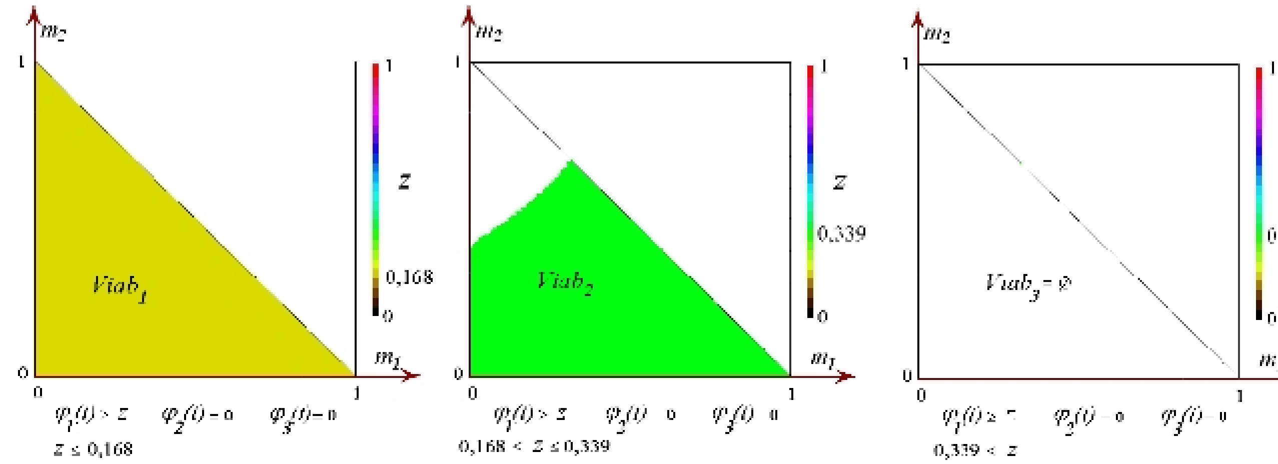
$$Viab(K) = K = \sum \times [0, C] \tag{7}$$

But we have to interpret this result with caution. It means that for an high C it always exists some $\varphi_h(t)$ which allow the economy to remain in K. Now it could be asked: is the monetary authority capable to implement such $\varphi_h(t)$? A further inquiry has to be done with constraints on the velocities of the $\varphi_h(t)'$s, that is on $\varphi'_h(t)$.

As sugested above, we have to check our intuition on the proportions. Namely, we have to examine whether an increase in credit money associated with a disproportion among the amounts of expenditures alters or not the size of the viability kernel.

The control constraint is now:

$$U = \left\{ \varphi \in \mathbb{R}^H \mid \varphi_h \geq 0, \sum \varphi_1 \geq z \text{ and } \varphi_2 = \varphi_2 = 0 \right\} \tag{8}$$



and K is (here $z = 0$):

$$K := \left\{ m \in \mathbb{R}^H \times \mathbb{R} \mid m_h \geq \ 0, \sum_h m_h = 1 \right\} \tag{9}$$

Numerical simulations confirm that less than the absolute amount of credit it is rather the disproportion between individual expenditures allowed by an important credit which matters for (in)stability. Figure below shows that two thresholds exist. Below the first (= 0.168) the viability kernel remains unchanged and equal to the simplex; above the second (0.339) the viability kernel is empty. The economy disappears or must change dramatically. In the range between the two thresholds the viability kernel is reduced but does not change:

References

[1] Aubin, Jean-Pierre, Bayen, Alexandre & Saint-Pierre, Patrick, (2011), *Viability Theory, New Directions*, Second edition, Springer.