Multidimensional Risk and Risk Dependence

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Abstract

Evaluating multiple sources of risk is an important problem with many applications in finance and economics. In practice this evaluation remains challenging. We propose a simple non-parametric framework with several economic and statistical applications. In an empirical study, we illustrate the flexibility of our technique by applying it to the evaluation of multidimensional density forecasts, multidimensional Value at Risk and dependence in risk.

Keywords: Multiple Sources of Risk; Multidimensional Value at Risk; Risk Distribution, Dependence in Risk; Systemic Risk.

JEL classification: C52; C53.

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1. Introduction

Most of the financial literature assumes that two considerations are of utmost importance for an investor: the reward that may be attainable and the inherent risk in obtaining this reward. The trade-off between reward and risk is the essence of any investment strategy. While it is straightforward to approximate the reward by the return on the investment, the definition of risk is more ambivalent since it involves quantifying various sources of uncertainty about the future investment value.

Conceptually, risk is the potential for (adverse) deviation from expected results. Different proxies for risk have been proposed, where perhaps the most popular in the univariate context is the variability of returns, as measured by the variance. If returns are not drawn from a normal distribution, then variance is no longer an appropriate measure of risk because it fails to capture some of the characteristics of the return distribution that investors consider important. An alternative univariate risk measure is the Value at Risk (VaR), which is defined as the maximum loss on a portfolio over a certain period of time that can be expected with a nominal probability. When returns are normally distributed, the VaR of a portfolio is a simple function of the variance of the portfolio (Szegö, 2002). However, when the return distribution is non-normal, as is now the general consensus, the VaR of a portfolio is determined not just by the portfolio variance but by the entire conditional density of returns, including skewness and kurtosis (see Tay and Wallis, 2000).

Risk management, generally, involves more than one risky asset and is particularly concerned with the evaluation and balancing of the impact of various risk factors. If the joint distribution of asset returns is multinormal, then the correlation coefficient adequately captures the dependence between assets (see Diebold et al., 1999). However, joint normality is not supported by empirical evidence (see, for example, Patton, 2004). Moreover, correlation is only a measure of linear dependence and suffers from a number of limitations (see Embrechts et al., 2002; Patton, 2004). These deficiencies are compounded in the covariance measure which is an explicit function of the individual variables variances and their correlation. The overreliance on covariances can have detrimental consequences as they are an essential input in many financial applications including hedging and portfolio decisions. Indeed, Embrechts et al. (2002) warn that unreliable risk management systems are being built using correlations – and by extension covariances – to model dependencies between highly non-normal risks.

While in the univariate context, the shortcomings of variance as a risk measure have been mostly addressed by VaR, the financial literature that explicitly addresses the shortcomings of the covariance as a measure of co-dependence is still in its infancy. Failure to properly characterize the relationships and inter-dependence of the multiple risk factors can have severe consequences as demonstrated by the recent failure of the rating agencies to account for house price risk when rating structured products (Gorton, 2010). Moreover, while the financial literature is replete with techniques which model the dependence in return (e.g. CAPM, APT), the equally important matter of the dependence in risk has only recently come to attention (see Patton, 2009).

This paper makes the following contributions to the nascent literature on multi-factor risk. Firstly, it proposes a simple and flexible statistical framework to evaluate time-varying, density forecasts of multidimensional risks. Secondly, VaR is generalized in a natural way. Essentially, multidimensional Value at Risk (MVaR) is a region of the intersection of univariate VaRs with a nominal probability mass under a given density function. It turns out that MVaR is a versatile framework that allows for examining and evaluating the dependence in risk. MVaR can also be seen as a straightforward illustration of the multiple sources of risk: If VaR is a univariate risk measure, which instead of the variance takes into account the entire tail density, then MVaR is a measure of multidimensional risk that instead of the covariances takes into account the entire distribution in the relevant joint tail.

The outline of the remainder of this paper is as follows. In Section 2, we present an economic motivation for MVaR, while in Section 3 we discuss the concept of joint density tails. In Section 4, we illustrate the application of this framework to multidimensional density forecasts (MDF) evaluation. Section 5 introduces MVaR and discusses its various statistical and economic interpretations, while Section 6 applies the MVaR framework to the measurement of the dependence in risk. Section 7 presents a small empirical study to illustrate these concepts. Finally, Section 8 concludes.

2. Motivation for Multidimensional Risk

Effective risk management requires not only correct identification of the sources of risk but also an adequate capturing of their distributional characteristics. Examples of the importance of properly accounting for the multiple sources of risk come from the financial economics literature. A major contribution to this literature, the Capital Asset Pricing Model (CAPM), models asset returns by decomposing their variability into market risk and firm-specific effects that can be diversified away in large portfolios. In this model, the return on the market portfolio summarises the broad impact of macroeconomic factors. However, often rather than using a market proxy, it is more enlightening to focus directly on the ultimate, individual sources of risk. This can be useful in risk assessment, when measuring exposures to particular sources of uncertainty. Arbitrage Pricing Theory (APT) shows how a decomposition of risk into systematic and idiosyncratic influences can be extended to deal with the multifaceted nature of systematic risk. Multifactor models can be used to measure and manage exposure to each of the multiple economy-wide risk factors such as, e.g. business-cycle risk, inflation, interest and exchange rate risk and energy price risk (see, for example, Ferson and Harvey, 1994; Chan et al., 1998).

The recent financial crisis brought to the forefront of attention systemic risk. This is the risk of collapse faced by the financial system as a whole when one of its constituent parts gets into financial distress. Due to the interconnectivity of the financial institutions, a shock faced by one institution in the form of an extreme event, increases the probability other financial institutions experiencing similar extreme events, leading to a domino effect (see Gai and Kapadia, 2010; Nijskens and Wagner, 2011). At the individual level, financial institutions are subject to three types of risk: market, credit and operational risk. For example, market risk typically generates portfolio value distributions that are often approximated as normal. Credit and especially operational risk generate more skewed distributions due to occasional extreme losses. For examples of market risk, see Jorion (2001). Crouhy et al. (2001) give examples of all three risk types while Kuritzkes et al. (2003) present stylized pictures of a very broad range of risk types that are faced by large financial companies. In recent years, there has been increasing concern among researchers, practitioners and regulators over the evaluation of models of financial risk. Moreover, while it is important to have an aggregate measure of the total risk, often it is also important to know the direct dependence on, and inter-relationships of, the specific market, credit and operational sources of risk. These developments accentuate the need for evaluation techniques that are flexible and yet powerful (Lopez and Saidenberg, 2000).

While the literature on aggregating the multiple sources of risk is recently gaining momentum (see, for example, Rosenberg and Schuermann, 2006), there appears to be virtually no research into the joint evaluation of such sources of risk or to characterize their inter-dependence. Moreover, while some risk types are more easily characterized and measured than others, much less is known about their joint behaviour, distributional characteristics and cross-influences (see, for example, Chollete et al., 2011). By focusing on the joint distribution of the individual sources of risks, we provide a framework to characterise the co-dependence of these risks. It is important to emphasize that such a framework is not merely statistically interesting. As demonstrated by the recent financial crisis, financial institutions and regulators are in fact concerned with the possibility that their risk models do not adequately describe tail events. Indeed, a type of model failure of particular interest to financial institutions and regulators is that in which the forecasted probabilities of large losses are inaccurate or worse, underestimated.[1]

3. Joint Density Tails

In this section, we introduce definitions that will be used throughout the paper. A joint density tail (JDT) is an unbounded region of the Euclidean space that is marked off by cut-off values. A parsimonious definition of the JDT $O(d, v)$ in the N-dimensional linear space R^N over the real line R requires only one cut-off value $v \in R$ and a directional vector $d \in R^N$ as illustrated in Figure 1,

$$O(d, v) := \{y \in R^N : y_i/d_i \geq v,\ \forall d_i \neq 0\} \tag{1}$$

[Figure 1]

For instance, if $N = 2, d = (1,0)$ and $v = 0$, then $O(d, v)$ is the half-plane containing all points with non-negative first coordinates. It follows directly from definition (1) that $O(d, v)$ is an intersection of univariate tails,

$$O(d, v) = \bigcap_{i:d_i \neq 0} O(d_i u^i, v)$$

[1] When the Federal Reserve Chairman Ben Bernanke was asked by the Financial Crisis Inquiry Commission what academic papers he recommends reading about the financial crisis and its aftermath, he suggested, among others, a paper by Adrian and Brunnermeier (2009), which proposes CoVaR ("Co" stands for conditional, contagion, or co-movement) as a way to measure a firm's systemic risk (see http://blogs.wsj.com/deals/2010/09/02/ben-bernankes-labor-day-reading-list/). CoVaR is a nested measure of our MVaR framework.

where u^i is the unit vector in direction $i = 1, \dots, N$ and $O(d_i u^i, v)$ is a half-hyperspace in R^N.

Given an N-dimensional probability density function (PDF) f, the probability mass of the JDT $O(d, v)$ under f is computed as

$$z^d(v, f) = \left| \int_{d_N v}^{d_N \infty} \dots \int_{d_k v}^{d_k \infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(\tau_1, \dots, \tau_N) d_{\tau_1} \dots d_{\tau_N} \right| \qquad (2)$$

where we first integrate over the entire real line for the variables $\tau_1, \dots \tau_{k-1}$ with $d_1 = \cdots = d_{k-1} = 0$. Given a line in R^N along the directional vector $d \in R^N$, we define the following projection x^d of the point $x \in R^N$ on this line,

$$x^d = v^d(x) \cdot d \text{ where } v^d(x) = \min_{d_i \neq 0} \{x_i / d_i\} \qquad (3)$$

The point x is projected along the axis i that corresponds to the minimum ratio x_i / d_i. The next proposition states that the projection x^d of a point x that lies inside (outside) of the JDT $O(d, v)$, stays inside (outside) this JDT.

Proposition 1: For a directional vector $d \in R^N$, $d \neq 0$, and $v \in R$,

$$x \in O(d, v) \Leftrightarrow x^d \in O(d, v) \qquad (4)$$

Figure 2 presents an intuitive proof of this proposition, while the formal proof is given in the Appendix.

[Figure 2]

The JDT definition (1) is of fundamental importance for our measures of MVaR, risk dependence and for risk forecasting, while Proposition 1 is crucial for a systematic evaluation of these concepts. Unlike other measures of multidimensional risk (e.g., Embrechts and Puccetti, 2006), we focus on risk along a pre-specified direction as expressed by the directional vector[2] d. The first advantage of this approach is that it reduces a multidimensional problem to a unidimensional one, which then allows for the application of the well-known techniques of risk evaluation. In particular, the univariate VaR will be obtained as a special case. Secondly, its flexibility means that risk can be evaluated over specific areas as defined by the user's interest. For example, an investor who is exposed to a subset of assets is more interested in the joint probability of extreme events involving these assets than in an overall measure of market risk. Thus, our model does not impose any constraints on the choice of the directional vector. This decision pertains to the user who tailors the directional vector according to their particular interest.

4. Multidimensional Density Forecast Evaluation

Our first application of Proposition 1 is the evaluation of time-varying multidimensional densities. Recently, the trend in the finance and economics literature is decisively towards

[2] The directional vector d has a distinct financial interpretation. For example, suppose a hypothetical investor holds a portfolio that is long two units in Asset 1 and short one unit in Asset 2. Then, a directional vector of particular interest for this investor is $d = (d_1, d_2)' = (2, -1)'$ as it succinctly represents his portfolio in R^2 space.

joint density forecasting.[3] The increasing importance of forecasts of the entire conditional joint density naturally raises the issue of forecast evaluation. The relevant literature, although developing at a fast pace, is still in its infancy. This is somewhat surprising considering that the crucial tools date back a few decades. Rosenblatt (1952) showed that for the cumulative distribution function CDF F_t (PDF f_t), which correctly forecasts the true data generating process (DGP) for the random variable X_t, the probability integral transformation (PIT) $Z_t = F_t(X_t)$ is *i.i.d.* $U[0,1]$ over t. Therefore, the adequacy of forecasts can be easily evaluated by examining the $\{z_t = F_t(x_t)\}_{t=1}^{T}$ series for violations of independence and uniformity.

The PIT idea is extended to the multivariate case by Diebold et al. (1999). Their test procedure factors each period's MDF into the product of the conditionals and obtains the PIT for each conditional distribution. As a result, this procedure generates a PIT series for each conditional and the generated series can be tested for violations of independence and uniformity individually and as a whole. Rejecting the null of *i.i.d.* $U[0,1]$ for any series implies that the MDF is misspecified (see also Clements and Smith, 2000, 2002). However, these approaches rely on the factorization of each period forecasts into their conditionals, which may be impractical for some applications, such as for numerical approximations of MDFs often employed in finance (see the survey of Corradi and Swanson, 2006).

[3] See, for example, the recent issue on copulas and multivariate distributions in the *European Journal of Finance* (2009), 15(7&8) and the references therein. See also the Special Issue on Density Forecasting in Economics and Finance in the *Journal of Forecasting* (2000), 19, 231–392.

Our Proposition 1 implies that the potentially cumbersome factorization of a MDF can be circumvented and uniformly distributed scores can be obtained by exploiting the properties of projection (3). Specifically, our JDT-test[4] uses the projection $x^d = v^d(x) \cdot d$ of the original observation $x \in R^N$ to compute the corresponding score as the probability mass (2) over the induced JDT $O(d, v^d(x))$, where $v^d(x) = \min_{d_i \neq 0}\{x_i/d_i\}$. In the Appendix, we prove the following result.

Corollary 1: If the sequence of time-varying CDFs $\{F_t\}_{t=1}^T$ is the true DGP for the sequence $\{X_t\}_{t=1}^T$ of N-dimensional random variables $X_t = (X_{1,t}, \dots, X_{N,t})$, then the z-scores $\{Z_t^d = F_t(v^d(X_t))\}_{t=1}^T$, where $v^d(X_t) = \min_{d_i \neq 0}\{X_{i,t}/d_i\}$ and $d \in R^N$, $d \neq 0$, are *i.i.d.* $U[0,1]$ over t.

Note that for unidimensional forecasts our procedure reduces to the traditional PIT. Importantly, the proposed procedure effectively transforms a CDF F_t into a unidimensional random score variable. Further, the corollary implies that the null hypothesis of the overall accuracy of a density model can be tested by the standard tests of uniformity (see Noceti et al., 2003) and independence (see Brock et al., 1991). When rejecting the null, we can dismiss the density model as inaccurate. Acceptance, on the other hand, means that the density specification is compatible with the sample, when verified along the directional vector $d \in R^N$. We discuss the issue of parameter estimation uncertainty in the Appendix.

[4] Due to the focus on the joint density tails, we call our procedure the JDT-test.

5. Multidimensional Value at Risk

The unidimensional Value at Risk (VaR) is now one of the most widely used measures of tail risk among practitioners, largely due to its adoption by the Basel Committee on Banking Regulation (1996) for the assessment of the risk of the proprietary trading books of banks and its use in setting risk capital requirements (see Gourieroux and Jasiak, 2010). For the unidimensional continuous CDF F_t (PDF f_t), the VaR at the nominal level a is the quantile v_a for which $F_t(v_a) = a$. From the VaR definition follows that the probability mass (2) under f_t of the interval $\{y \in R: y \leq v_a\}$ is equal to the nominal level a.

In analogy to VaR, we define the multidimensional Value at Risk in direction $d \in R^N$ at the nominal level a ($MVaR_a^d$) as the cut-off value $v^d(f,a) \in R$ such that the probability mass (2) under f of the JDT $O(d, v^d(f,a))$ is equal to the nominal level a. However, we will often identify the $MVaR_a^d$-value $v^d(f,a)$ also with the JDT $O(d, v^d(f,a))$. Then, depending on the context, we will refer to $MVaR_a^d$ either as a (probabilistic) event or as a set. The boundary of the latter set in direction $i = 1, \dots, N$ is defined by the value $v^d(f,a) \cdot d_i$, where $d_i \neq 0$. We will say that $x \in R^N$ is an extreme observation, whenever the projection $v^d(x)$ of x exceeds (violates) the threshold $v^d(f,a)$ or, equivalently, when x falls into the set $MVaR_a^d$,

$$v^d(x) \geq v^d(f,a) \Leftrightarrow x \in MVaR_a^d \qquad (5)$$

Note that the equivalence of these two events follows from Proposition 1. Due to the decomposition of JDTs into univariate tails, $MVaR_a^d$-set can be seen as an intersection of univariate VaRs, as illustrated in Figure 3 for a bivariate distribution.

[Figure 3]

It is important to note that our MVaR definition differs in important ways from the MVaR measure introduced in Embrechts and Puccetti (2006). They define the multivariate lower-orthant (LO-)VaR as $\underline{VaR}_a(F) \coloneqq \partial\{x \in R^N : F(x) \leq a\}$. In spite of its mathematical appeal, their measure is of limited use for our purposes. Most importantly, the probability under the MDF F of the region $\{x \in R^N : F(x) \leq a\}$, i.e., where the LO-VaR is exceeded, is generally not equal to a when $N > 1$. Therefore, it is not clear how the LO-VaR can be evaluated in empirical applications. Further, and unlike MVaR defined in this paper, it is not clear how LOVaR can be adapted to measure risk dependence (see Section 6 below). Similar observations apply to the second multivariate risk measure, introduced in Embrechts and Puccetti (2006), the upper-orthant (UO-)VaR.

In spite of their conceptual simplicity, working directly with MVaRs can prove challenging in higher dimensions. However, we will show that the relevant MVaR inference can be more easily obtained from the z_t^d-scores. Specifically, for the projection $x^d = v^d(x) \cdot d$ of observation x, we will compute its z_t^d-score as the probability mass (2) under f of the corresponding JDT $O(d, v^d(x))$. Our next result (proved in the Appendix) characterizes the score values of observations in the $MVaR_a^d$.

Corollary 2: For a continuous PDF f, a directional vector $d \in R^N$, $d \neq 0$, and a nominal significance level $a \in (0,1)$,

$$x \in MVaR_a^d \Leftrightarrow z^d(v^d(x), f) \leq a \tag{6}$$

An important application of Corollary 2 is the evaluation of MVaR forecasts for a given sequence of time-varying density forecasts $\{\hat{f}_{t-1}\}_{t=1}^T$ based on a sequence of multidimensional observations $\{x_t\}_{t=1}^T$. Corollary 2 implies that, under the correct forecasting model, the proportion of z_t^d-scores, computed for $\{x_t\}_{t=1}^T$ and with values below a, should approach the nominal significance level a for a sufficiently large sample. We refer to this procedure as unconditional accuracy. On the other hand, the conditional accuracy requires that the number of scores with values less than a should be unpredictable when conditioned on the available information. In other words, the $MVaR_a^d$ violations (i.e. extreme observations whose projections along the vector d exceed the $MVaR_a^d$ threshold) should be serially uncorrelated. Thus, if the forecasting model is correct, an extreme observation today should contain no information as to whether an extreme observation will occur tomorrow. To assess both types of accuracy, we can resort to the unconditional accuracy test of Kupiec (1995) and the conditional accuracy test of Christoffersen (1998). Although both tests are designed for testing the univariate VaR accuracy, they still apply for our purposes because the score computation effectively converts a MDF into a univariate score variable.

5.1 The Interpretation of MVaR as a Risk Distribution Measure

In the Introduction, we noted that variance is an inadequate risk measure if the return distribution is non-normal. If this is the case, then an adequate risk measure must incorporate all the characteristics that define the return distribution. As such, VaR addresses the shortcomings of the variance as it incorporates the entire tail distribution. In analogy, MVaR can be interpreted as a measure of multidimensional risk that takes into account the entire density in a JDT. Further, MVaR leads naturally to a risk distribution function (RDF). An RDF can be constructed for N risk factors $Y_1, \dots, Y_N$ with the joint PDF f when the payoff of an agent depends on the standardized factors $Y_1/d_1, \dots, Y_N/d_N$. The standardization can account for, e.g., different units or variances of the risk factors. Then, the following definition of the RDF for this agent,

$$\Psi(v;d) = Pr_f(O(d,-v)) \text{ where } O(d,-v) := \{y \in R^N: y_i/d_i \geq -v,\ \forall d_i \neq 0\}$$

satisfies the conditions of a CDF: $\Psi(-\infty;d) = 0$, $\Psi(\infty;d) = 1$ and $\Psi(v;d)$ non-decreasing in the first argument. The RDF $\Psi(v;d)$ is the probability that all standardized risk factors exceed the critical level v simultaneously. For an observation x, we can also compute the cut-off value $v^d(x)$. If the latter value exceeds (or violates) $v^d(f,a)$, a risk event at significance level a has occurred.

6. Dependence in Risk

In this section, we define two risk dependence measures that rely on the MVaR framework. The first one is the relative change in the probability of $MVaR_a^d$, given that

$MVaR_{\tilde{a}}^{\tilde{d}}$, $d \neq \tilde{d}$, has occurred. Specifically, for the multidimensional random variable R with the joint PDF f,

$$p_{\tilde{a}}^{a}(f,d,\tilde{d}) \coloneqq Pr_f\left(R \in MVaR_a^d \middle| R \in MVaR_{\tilde{a}}^{\tilde{d}}\right) = \frac{Pr_f(MVaR_a^d \cap MVaR_{\tilde{a}}^{\tilde{d}})}{Pr_f(MVaR_{\tilde{a}}^{\tilde{d}})}$$

is the conditional probability of the $MVaR_a^d$-event, given the occurrence of $MVaR_{\tilde{a}}^{\tilde{d}}$. By the definition of statistical independence, it holds in the special case $a = \tilde{a}$ that,

$$p_a^a(f,d,\tilde{d}) = Pr_f(MVaR_a^d) = a$$

when the events $MVaR_a^d$ and $MVaR_a^{\tilde{d}}$ are independent. Therefore, we can express the degree of dependence between these events by the relative change in conditional probability,

$$\gamma_a(f,d,\tilde{d}) = \gamma_a(f,\tilde{d},d) \coloneqq \left(p_a(f,d,\tilde{d}) - a\right)/a \tag{7}$$

which is equal to zero when $MVaR_a^d$ and $MVaR_a^{\tilde{d}}$ are independent.[5] Positive values of $\gamma_a(f,d,\tilde{d})$ indicate that the occurrence of $MVaR_a^{\tilde{d}}$ increases the probability of $MVaR_a^d$, while negative values indicate the opposite. Note that changes in the nominal level a do not affect $\gamma_a(f,d,\tilde{d})$ as long as the ratio $p_a(f,d,\tilde{d})/a$ remains constant,

[5] An alternative dependence measure can be defined as

$$\hat{\gamma}_a(f,d,\tilde{d}) \coloneqq \left(p_a(f,d,\tilde{d}) - a\right)/\left(p_a(f,d,\tilde{d}) + a\right)$$

The benefit of this measure is that it is normalized to lie between -1 and 1, while it attains the value of zero when $MVaR_a^d$ and $MVaR_a^{\tilde{d}}$ are independent.

$$p_a\ (f,d,\tilde{d}\ )/a = p_{\tilde{a}}\ (f,d,\tilde{d}\ )/\tilde{a} \Rightarrow \gamma_a\ (f,d,\tilde{d}\ ) = \gamma_{\tilde{a}}\ \ (f,d,\tilde{d})$$

Our second dependence in risk measure, conditional MVaR (CMVaR), is similar to CoVaR in Adrian and Brunnermeier (2009) and is defined as the relative change in the $MVaR_a^d$-value when conditioned on the $MVaR_a^{\tilde{d}}$-event,

$$CMVaR_a^{d,\tilde{d}} = \left(v^d\left(f\middle|MVaR_a^{\tilde{d}},a\right) - v^d(f,a)\right)/|v^d(f,a)| \qquad (8)$$

where $v^d(f,a)$ is computed with respect to the PDF f while $v^d(f|MVaR_a^{\tilde{d}},a)$ is computed with respect to the conditional PDF $f|MVaR_a^{\tilde{d}}$, i.e., with respect to the normalized density f over $MVaR_a^{\tilde{d}}$. This measure indicates the relative change in the $MVaR_a^d$-value, when conditioned on the occurrence of $MVaR_a^{\tilde{d}}$. If the latter event has no impact on $MVaR_a^d$, then $CMVaR_a^{d,\tilde{d}}$ is equal to zero. On the other hand, positive (negative) values of $CMVaR_a^{d,\tilde{d}}$ indicate that the conditioning increases (decreases) the risk, as measured by $MVaR_a^d$. Note that CMVaR can also be employed to measure systemic risk and contagion. For example, if MVaR-value $v^d(f,a)$ measures the unconditional risk of a financial system, then $CMVaR_a^{d,\tilde{d}}$ may capture the exposure of this system to an institution, represented by $\tilde{d}$, experiencing the extreme event $MVaR_a^{\tilde{d}}$.

The dependence measures can be computed either from a theoretical density function f or from observations that define an empirical distribution f_E. The task is simplified in the latter case as multidimensional integration in the calculation of MVaR-values from the

PDF f is replaced by the computation of the corresponding quantiles for unidimensional projections (3) of observations in f_E along directional vectors.[6]

For example, in order to compute $Pr_{f_E}(R \in MVaR_a^d | R \in MVaR_{\tilde{a}}^{\tilde{d}})$, we first select the $MVaR_a^d$ ($MVaR_{\tilde{a}}^{\tilde{d}}$) to contain the proportion a ($\tilde{a}$) of observations in f_E with the largest projections on the line along the directional vector d ($\tilde{d}$). Then, we compute the empirical conditional probability from the number of observations in the intersection $MVaR_a^d \cap MVaR_{\tilde{a}}^{\tilde{d}}$ over the number of observations in $MVaR_{\tilde{a}}^{\tilde{d}}$. We can compute $CMVaR_a^{d,\tilde{d}}$ in a similar manner. Importantly, these computations can be performed efficiently in higher dimensions and for large samples. Therefore, $\gamma_a(f, d, \tilde{d})$ and $CMVaR_a^{d,\tilde{d}}$ are convenient and robust non-parametric tools for analyzing dependence in multidimensional data.

7. Empirical Illustration of the MVaR Framework

In this section, we present a small empirical study to illustrate the evaluation and measurement of the multiple sources of risk in the proposed framework. We employed three-dimensional observations composed of the daily log returns for the S&P 500 stock index, the spot index of a basket of commodities computed by the Commodities Research Bureau CRB and the GBP/USD exchange rate (SP, CI and DP hereafter). The data were obtained from Datastream[7] and cover the period 3 January 1972 to 14 September 2010 10097 synchronized daily observations.

[6] Note that the observation x_1 is more extreme (in direction d) than the observation x_2 if and only if $v^d(x_1) > v^d(x_2)$.

[7] Datastream provides global financial and macroeconomic data and is owned by Thomson Reuters. For details see http://thomsonreuters.com/

We interpret the three variables as representing different sources of risk originating in the domestic, commodities and foreign exchange markets respectively. Table 1 presents summary statistics for the continuously compounded daily returns for the three synchronized time series. All mean returns are close to zero and there is a weak positive correlation between the factors. In line with previous evidence, the distribution of daily returns is heavily leptokurtic.

[Table 1]

In the first part of the experiment, we test the accuracy of two parametric distributions – the multinormal (MN) and the multivariate t-distribution (MT) – and the accuracy of the adaptive empirical distribution (AED). All specifications are time-varying. The details of the dynamic estimation of the parametric distributions via a multivariate GARCH model are given in the Appendix.

We use the observations 3001 to 10097 to compute the z_t^d-scores for all three specifications (the first 3000 observations were used for the initial estimation). The z_t^d-scores are computed in directions that are proportional to the standard deviations σ_{SP}, σ_{CI} and σ_{DP} of the respective sample returns. For example, $MVaR_a^d$ in direction $d = (\sigma_{SP,}\sigma_{CI}, \sigma_{DP})$ contains all observations whose projections exceed $v^d(f, a)$, where f is either the MN, MT or the AED and a is the nominal significance level. Although our choice of the directional vector serves only as an illustration of the techniques involved, we made it proportional to the empirical standard deviations of the risk sources for the sake of normalization and, hence, comparability of these risks.

The results of the experiment are reported in Table 2. First, we focus on the overall accuracy of the three specifications by computing the p-values of the uniformity test for the z-scores (numbers in bold). For all distributions and along all tested directions, the p-values are essentially equal to zero, which implies that the data is not generated from any of the specifications. Note that for the directional vectors $(\sigma_{SP}, 0,0)$ and $-(0, \sigma_{CI}, 0)$, we effectively test the null that the factors SP and CI were generated by the respective marginals of MN, MT and AED.

[Table 2]

Comparing the MVaR accuracy of the different models reveals that at low significance levels only the MT can approximate the true DGP. Given the estimate of 2.7 for the degrees of freedom i.e. the relatively thick tails, this finding confirms the universality of the approximately cubic law for extreme returns (see Gopikrishnan et al., 1998) in the multivariate context. On the other hand, AED is the model of choice at high significance levels. Interestingly, the example of the directional vector $-(\sigma_{SP}, \sigma_{CI}, \sigma_{DP})$ shows that extreme simultaneous decreases cannot be estimated reliably from past observations. More precisely, the probabilities of such extreme events will be consistently underestimated – a potentially severe problem for risk management – as illustrated by the fact that the AED leads to the proportion of z-scores with frequencies of observations less than or equal to a that is far above this nominal level a. This finding suggests that the probability of all negative extreme events has increased steadily since 1972 i.e., during our sampling period.

The second part of our empirical study focuses on MVaR risk dependence measures and uses the same observations on S&P 500 index, CRB index and the GBP/USD exchange rate as the accuracy tests. We refer to this data as Sample A. In addition, we compute the dependence measures for synthetic data (Sample B) that was drawn from a three-dimensional multinormal with the same parameters as Sample A. In Table 3, we report for each nominal level a and vectors d and $\tilde{d}$, the dependence coefficient, CMVaR and the tail correlation. The latter is computed as the correlation coefficient for observations in the relevant joint density tail. It quantifies the *linear* dependence of variables in this tail (see Longin and Solnik, 2001).

[Table 3]

We observe that the CMVaR decreases in a in Sample A while it tends to increase in the synthetic Sample B, although the former CMVaR is always significantly higher than the latter. In particular, for the directional vectors $d = -(0, \sigma_{CI}, 0)$, $\tilde{d} = -(\sigma_{SP}, 0,0)$ and a =0.05, the conditional MVaR-value increases in absolute value by 108.9% in Sample A, while it increases only by 7.8% in Sample B. The different patterns for the two samples are quite pronounced. In particular, the highest CMVaR for Sample B lies below the lowest CMVaR for Sample A. Therefore, the risk dependence for market returns differs significantly from that corresponding to the multinormal synthetic data.

Further, we observed for Sample A that the tail correlation decreases in the "positive" joint tail, i.e., for $d = (\sigma_{SP}, 0{,}0)$ and $\tilde{d} = (0, \sigma_{CI}, 0)$, while it increases in the "negative" tail $d = -(0, \sigma_{CI}, 0)$ and $\tilde{d} = -(\sigma_{SP}, 0{,}0)$. Both joint tails are bidimensional as they are the intersections of univariate tails. In the three-dimensional "negative" tail, defined by $d = -(0{,}0, \sigma_{DP})$ and $\tilde{d} = -(\sigma_{SP}, \sigma_{CI}, 0)$, the correlation coefficients between S&P 500 and CRB index and between S&P 500 and the GBP/ USD exchange rate were typically higher than the coefficients in the bidimensional tails. Interestingly, these coefficients fall in a except for a = 0.01 contrasting with the trend in the bidimensional "negative" tail. The tail correlation in Sample B is typically lower than in Sample A, lying below 10%, with the exception of 1% significance. Finally, the dependence coefficient decreases in a for both samples and all directional vectors.

Although the results for a = 0.01 seem to contradict the general trend in some cases, their significance is limited as they were computed from small sets of observations. Generally, if the probabilities of the event A_1 and the event A_2 are a and these events are independent, then the expected proportion of observations in the intersection $A_1 \cap A_2$ is a^2. Specifically, for our sample of ca. 10,000 observations, a=0.01 and under the independence of $MVaR_a^d$ and $MVaR_a^{\tilde{d}}$, we expect $10000 \cdot 0.01^2 = 1$ observation in the joint tail $MVaR_a^d \cap MVaR_a^{\tilde{d}}$.

Figure 4 illustrates succinctly the co-dependence between the risk factors S&P 500 index, CRB index and GBP/USD exchange rate. The interdependence appears to be particularly strong at the 1% significance level although our caveat of small samples applies also in this case. Hence, we focus on the results for the 10% significance level. Whilst the dependence coefficient between $MVaR_a^d$ and $MVaR_a^{\tilde{d}}$ does not depend, by definition, on the order of directional vectors, i.e., $\gamma_a(f_E, d, \tilde{d}) = \gamma_a(f_E, \tilde{d}, d)$, the conditional MVaRs may depend on this order. For example, the probability of exceeding the 10% SP-VaR (10% DP-VaR) increases by 25.7% when the 10% DP-VaR (10% SP-VaR) has been exceeded. On the other hand, the conditional 10% SP-VaR (conditional 10% DP-VaR) increases by 13% (17%) when the 10% DP-VaR (10% SP-VaR) has been exceeded. This difference in conditional VaRs indicates an asymmetric reaction to negative shocks. Note that the same framework can be applied to measure systemic risk e.g., by using S&P 500 index as the "system" and a particular stock as a risk factor.

[Figure 4]

Finally, Table 4 summarizes our results on intertemporal risk dependence for the time-lagged log returns (r_t, r_{t+1}, r_{t+2}) on the S&P 500 index. Although the three series, computed from the same data as in the previous experiments, are essentially uncorrelated, there is a strong dependence in the tails. For example, the probability that the return tomorrow r_{t+1} will fall below the unconditional 10% VaR is 63% higher, when the return today r_t fell below the unconditional 10% VaR. The conditional MVaR, on the other hand, increases by 50.5% in absolute value, when conditioned on the same event. The tail dependence tends to decrease with higher a in accordance with our previous results. Therefore, the temporal clustering of risk appears to be stronger for more extreme events.

[Table 4]

8. Conclusions

The focus of the financial literature has recently shifted to aggregating the multiple sources of risk. While interesting in their own right, such approaches cannot comprehensively answer questions which are paramount for pricing, hedging and portfolio decisions. Interesting answers can be obtained by considering the individual sources of risks jointly. We propose a simple and flexible framework with several statistical end economic applications. This framework allows for an evaluation of joint density forecasts, which is relatively straightforward even in higher dimensions. Moreover, it also allows for an evaluation of such densities over specific areas as defined by the user's interest. This is particularly convenient considering recent interest in the modelling of extreme events. A straightforward application of the proposed framework is the evaluation of the multidimensional Value at Risk and the measurement of dependence in risk. We illustrate such an application for a set of financial data in Section 7.

Finally, the data-intensive requirements of the MVaR framework make it natural for employing high-frequency returns to examine their risk dependence. It would also be interesting to investigate the potential of the MVaR framework for minimum-VaR portfolio decisions. We intend to pursue both these avenues in future research.

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Appendix A – Proofs

Proof of Proposition 1:

$$\Rightarrow: \quad x \in O(d,v) \Rightarrow x_i/d_i \geq v,\ \forall d_i \neq 0 \Rightarrow \min_{i:d_i\neq 0}\{x_i/d_i\} = v^d(x) \geq v$$
$$\Rightarrow v^d(x)d_i/d_i = x_i^d/d_i \geq v,\ \ \forall d_i \neq 0 \Rightarrow x^d \in O(d,v).$$

$$\Leftarrow: \quad x^d \in O(d,v) \Rightarrow x_i^d/d_i \geq v \Rightarrow\ v^d(x)d_i/d_i \geq v,\ \ \forall d_i \neq 0$$
$$\Rightarrow\ \min_{i:d_i\neq 0}\{x_i/d_i\} \geq v \Rightarrow x_i/d_i \geq v,\quad \forall i: d_i \neq 0 \Rightarrow\ x \in O(d,v),$$

where $x^d = v^d(x) \cdot d,\ v^d(x) = \min_{d_i \neq 0}\{x_i/d_i\}$ were defined in (3).

Proof of Corollary 1:

Uniformity on [0,1] follows from Corollary 2 and from the fact that $Pr_f(MVaR_a^d)$ = $Pr_f(v^d(X) \geq v^d(f,a))$ = a for any $a \in [0,1]$, where f (F) is the PDF (CDF) of the DGP for X. As the uniformity holds for any CDF F_t, the distribution of the scores at date t is $U[0,1]$ and independent of the distribution of the scores at any other date $s \neq t$.

Proof of Corollary 2:

$$\Rightarrow: \quad x \in MVaR_a^d \Rightarrow x_i/d_i \geq v^d(f,a),\ \forall i: d_i \neq 0 \Rightarrow$$
$$\min_{i:d_i\neq 0}(x_i/d_i) = v^d(x) \geq v^d(f,a) \Rightarrow z^d(v^d(x),f) \leq z^d(v^d(f,a),f) = a$$

$$\Leftarrow: \quad z^d(v^d(x),f) \leq a = z^d(v^d(f,a),f) \Rightarrow v^d(x) = \min_{i:d_i\neq 0}\{x_i/d_i\} \geq v^d(f,a) \Rightarrow$$
$$x_i/d_i \geq v^d(f,a),\ \forall i: d_i \neq 0 \Rightarrow x \in MVaR_a^d,$$

where $v^d(x) = \min_{d_i \neq 0}\{x_i/d_i\}$ was defined in (3).

Appendix B – Multivariate GARCH model and Estimation Method

To obtain forecasts of the time-varying three-dimensional covariance matrix we employ the simplified GARCH (S-GARCH) model of Harris et al. (2007). Note that for our parametric specifications MN and MT distribution, covariance matrix together with the degrees of freedom and means fully define the PDF. Our choice can be explained on the basis of the ability of this model to handle t-distributed residuals and its ease of estimation. The S-GARCH involves the estimation of only univariate GARCH models, both for the individual return series and for the sum and difference of each pair of series. The covariance between each pair of return series is then imputed from these conditional variance estimates. First, the conditional variances are estimated using univariate GARCH(1, 1) models:

$$r_{i,t} = \mu_i + \varepsilon_{i,t}, \qquad i = SP, CI, DP \tag{A1}$$

$$\sigma^2_{i,t} = a_{i,0} + a_{i,1}\sigma^2_{i,t-1} + a_{i,2}{\varepsilon^2}_{i,t-1}, \qquad i = SP, CI, DP \tag{A2}$$

Then six auxiliary variables, $r_{+,t} = r_{i,t} + r_{j,t}$ and $r_{-,t} = r_{i,t} - r_{j,t}$, $i, j = SP, CI, DP, i \neq j$ are constructed, and univariate GARCH(1, 1) models used to estimate the conditional variances of these.

$$r_{l,t} = \mu_l + \varepsilon_{l,t}, \qquad l = +, - \tag{A3}$$

$$\sigma^2_{l,t} = a_{l,0} + a_{l,1}\sigma^2_{l,t-1} + a_{l,2}{\varepsilon^2}_{l,t-1}, \qquad l = +, - \tag{A4}$$

where the residuals ε_i $i = SP, CI, DP, +, -$ are either normal or t-distributed. The conditional covariance between each pair of currencies is then imputed using the identity

$$\sigma_{ij,t} \equiv (1/4)(\sigma^2_{+,t} - \sigma^2_{-,t}) \tag{A5}$$

Like many other multivariate GARCH models, the S-GARCH does not guarantee that the conditional variance-covariance matrix is positive semi-definite. However, for all three pairs, the estimated correlation coefficients were found to be between –1 and +1 for all observations. We ML-estimate the S-GARCH parameters and the degrees of freedom parameter for the t-distributed residuals using the entire sample. We then use these estimates to obtain the forecast of the covariance matrix for $t = 3001, \dots, 10097$. On the other hand, the AED at time $t = 3001, \dots, 10097$, was defined by the sequence of the most recent 3000 observations. Note that all three MDFs vary over time.

Appendix C – Parameter Estimation Uncertainty

It is well known that the presence of estimated parameters may complicate test inference. For example, the Kolmogorov test can be difficult to apply in the presence of estimated parameters, particularly for multivariate data with many parameters (see, for example, Bai and Chen, 2008). Following other scholars (Diebold and Mariano, 1995; Christoffersen, 1998; Diebold et al. 1998, 1999; Clements and Smith, 2000, 2002), we consider the forecasts as primitives and ignore the method employed to obtain them. In many situations this may be an acceptable practice. Firstly, many density forecasts are not based on estimated models. For example, the large-scale market risk models at many financial institutions combine estimated parameters, calibrated parameters and ad-hoc modifications that reflect the judgment of management. Another example is the density forecasts of inflation of the Survey of Professional Forecasters (see Diebold et al., 1998). Moreover, previous research suggests that parameter estimation uncertainty is of second-order importance when compared to other sources of inaccuracies such as model misspecification (Chatfield, 1993). Further, Diebold et al. (1998) find that the effects of parameter estimation uncertainty are immaterial in simulation studies geared toward the relatively large sample sizes employed in financial studies such as the present one.

When parameter estimation cannot be ignored, the problem can be approached as follows. Firstly, for time-invariant forecasts, suitable estimators can often be found that lead to pivotal test statistics e.g., the "super-efficient" estimators (see Watson, 1958; Birch, 1964). Secondly, an important class of models comprises a time-varying hypothesised distribution with a well-defined structure on the co-evolution of the variables e.g. VAR and GARCH models. In this case, one way of accounting for parameter estimation uncertainty is to apply the *K*-transformation (Khmaladze, 1981), which allows for the construction of a distribution-free test statistic. In principle, the *K*-transformation can be applied to the JDT-test along the lines of the *V*-test in Bai (2003) and Bai and Chen (2008). Its computation, however, may be cumbersome for non-standard MDFs. Finally, in the case of arbitrary time-varying MDFs – for which our general model is particularly suited – parameter estimation is infeasible as only one observation is drawn from the MDF at each date. As such, the only practical solution is to assume that the hypothesised model is correct under the null.

Table 1: Synchronized Daily Returns

	S&P 500	CRB Spot Index	USD/GBP
Mean %	0.023	0.014	-0.004
Stand Dev	1.082	0.444	0.603
Skewness	-1.086	0.332	-0.095
Kurtosis	27.523	17.532	5.029
B-J	12070.545	12949.687	10658.712
ARCH4	692.510	703.646	798.691

Correlations

	S&P 500	CRB	USD/GBP
S&P 500	1.000	0.075	0.015
CRB		1.000	0.114
USD/GBP			1.000

Notes: The table reports the mean, standard deviation, skewness, excess kurtosis, Bera-Jarque statistic, ARCH4 statistic and the correlation matrix for the synchronized log returns for S&P 500, CRB Spot Index and the USD/GBP exchange rate for the sample period from 3 January 1972 to 14 September 2010 (10097 synchronized daily observations).

Table 2: Overall and directional accuracy of MN, MT and AED distribution

Nom. Sign./d	$-(\sigma_{SP},\sigma_{CI},\sigma_{DP})$	$(\sigma_{SP},\sigma_{CI},\sigma_{DP})$	$(\sigma_{SP},0{,}0)$	$-(0,\sigma_{CI},0)$
MN	**p-val. 0**	**p-val. 0**	**p-val. 0**	**p-val. 0**
$\alpha = 0.01$	0.017 (0)	0.016 (0)	0.015 (0)	0.014 (0)
$\alpha = 0.05$	0.043 (0)	0.040 (0)	0.039 (0)	0.042 (0)
$\alpha = 0.10$	0.063 (0)	0.071 (0)	0.071 (0)	0.074 (0)
$\alpha = 0.25$	0.346 (0)	0.334 (0)	0.180 (0)	0.184 (0)
MT(2.7)	**p-val. 0**	**p-val. 0**	**p-val. 0**	**p-val. 0**
$\alpha = 0.01$	0.010 (0.763	0.010 (0.689)	0.008 (0.021)	0.011 (0.449)
$\alpha = 0.05$	0.049 (0.231)	0.048 (0.076)	0.052 (0.156)	0.048 (0.053)
$\alpha = 0.10$	0.074 (0)	0.086 (0)	0.108 (0.01)	0.086 (0)
$\alpha = 0.25$	0.192 (0)	0.186 (0)	0.211 (0.0)	0.224 (0)
AED	**p-val. 0**	**p-val. 0**	**p-val. 0**	**p-val. 0**
$\alpha = 0.01$	0.012 (0.03)	0.014 (0)	0.013 (0.01)	0.013 (0.01)
$\alpha = 0.05$	0.055 (0.02)	0.057 (0)	0.055 (0.02)	0.056 (0.01)
$\alpha = 0.10$	0.095 (0.17)	0.095 (0.14)	0.105 (0.13)	0.114 (0)
$\alpha = 0.25$	0.245 (0.34)	0.250 (0.87)	0.253 (0.41)	0.255 (0.27)

Notes: The table reports the p-values of the χ^2-test of uniformity for the z-scores (bold), the exception rates i.e., the proportion of times the forecasted MVaR is exceeded and the p-values of the Kupiec statistic in parentheses. Sample period: from 4 July 1983 to 14 September 2010 (7097 synchronized daily observations). The degrees of freedom 2.7 for the MT distribution were ML estimated, given sample means and covariances.

Table 3: Risk Dependence for S&P 500, CRB Index and USD/GBP exchange rate

	$d=(\sigma_{SP},0,0)$ $\tilde{d}=(0,\sigma_{CI},0)$			$d=-(0,\sigma_{CI},0)$ $\tilde{d}=-(\sigma_{SP},0,0)$			$d=-(0,0,\sigma_{DP})$ $\tilde{d}=-(\sigma_{SP},\sigma_{CI},0)$		
Nominal Probab.	JDT Correl.	Dep. Coeff.	Cond. MVaR	JDT Correl.	Dep. Coeff.	Cond. MVaR	JDT Correl.	Dep. Coeff.	Cond. MVaR
				Sample A					
$\alpha=0.01$	.352	0.495	2.851	.012	0.866	1.654	.34/.08	0.844	1.232
$\alpha=0.05$	.193	0.291	0.490	.123	0.434	1.089	.58/.28	0.354	0.724
$\alpha=0.10$	.101	0.151	0.182	.225	0.265	0.483	.45/.20	0.223	0.347
$\alpha=0.25$	.081	0.067	0.148	.243	0.086	0.461	.27/.21	0.046	0.164
				Sample B					
$\alpha=0.01$	.061	0.400	0.077	-.144	0.600	0.099	-.22,-.1	0.900	0.097
$\alpha=0.05$	-.012	0.250	0.067	.043	0.310	0.078	0,.01	0.490	0.102
$\alpha=0.10$	-.023	0.240	0.098	.012	0.250	0.106	-.05,0	0.280	0.114
$\alpha=0.25$	.021	0.120	0.127	.020	0.110	0.132	0,.04	0.130	0.141

Notes: The table reports tail correlation, dependence coefficient $\gamma_a(f_E, d, \tilde{d})$ and conditional MVaR $CMVaR_a^{d,\tilde{d}}$ for the empirical distribution function f_E, nominal significance levels a= 1%, 5%, 10%, 25% and directional vectors d and $\tilde{d}$.
Sample A consists of synchronized log returns for S&P 500 (SP), CRB Spot Index (CI) and the USD/GBP exchange rate (DP) with mean $\mu=(\mu_{SP},\mu_{CI},\mu_{DP})$ and standard deviation $\sigma=(\sigma_{SP},\sigma_{CI},\sigma_{DP})$ as reported in Table 1. Observations were obtained from Datastream and cover the period from 3 January 1972 to 14 September 2010 (10,097) synchronized daily observations.
Sample B consists of 100,000 observations that were generated from the multinormal $N(\mu,\Sigma)$ where$\mu=(\mu_{SP},\mu_{CI},\mu_{DP})$ and Σ is the empirical covariance matrix computed from Sample A.

Table 4: Intertemporal Dependence for S&P 500

Nominal Probab.	$d = (\sigma, 0,0)$ $\tilde{d} = (0, \sigma_{t+1}, 0)$ JDT Correl.	Dep. Coeff.	Cond. MVaR	$d = -(0, \sigma_{t+1}, 0)$ $\tilde{d} = -(\sigma_t, 0,0)$ JDT Correl.	Dep. Coeff.	Cond. MVaR	$d = -(0,0, \sigma_{t+2})$ $\tilde{d} = -(\sigma_t, \sigma_{t+1}, 0)$ JDT Correl.	Dep. Coeff.	Cond. MVaR
$\alpha = 0.01$	.64	4.94	1.155	.47	7.91	6.808	-.16,.01	6.92	6.808
$\alpha = 0.05$	.54	0.98	0.447	.49	1.53	0.637	.09,.32	0.94	0.491
$\alpha = 0.10$	.43	0.43	0.247	.29	0.63	0.505	.20,.35	0.34	0.250
$\alpha = 0.25$	.24	0.03	0.042	.31	0.17	0.314	.14,.33	0.11	0.184

Notes: The table reports tail correlation, dependence coefficient $\gamma_a(f_E, d, \tilde{d})$ and conditional MVaR $CMVaR_a^{d,\tilde{d}}$ for the empirical distribution function f_E , nominal significance levels a= 1%, 5%, 10%, 25% and directional vectors d and $\tilde{d}$.
The sample was obtained from Datastream and consisted of S&P 500 log returns (r_t, r_{t+1}, r_{t+2}) for t=1,…,10,095 (3 January 1972 to 10 September 2010) with means $(\mu_t, \mu_{t+1}, \mu_{t+2})$ = (0.0236,0.0237,0.0237), standard deviations $(\sigma_t, \sigma_{t+1}, \sigma_{t+2})$ = (1.08299, 1.08303, 1.08302) and the correlation matrix ((1,0.004,-0.039), (0.004,1,0.004), (-0.039, 0.004,1)).

Figure 1: Directed line $\boldsymbol{\upsilon} \cdot \boldsymbol{d}$ and a JDT $\boldsymbol{O}(\boldsymbol{d}, \boldsymbol{\upsilon}_{\boldsymbol{a}})$ in $\boldsymbol{R}^{\boldsymbol{2}}$.

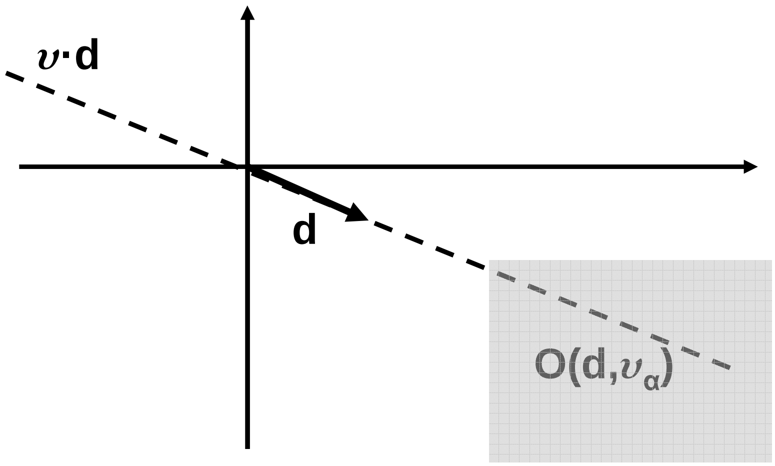


Figure 2: Projections on the directed line $\upsilon \cdot d$ in R^2.

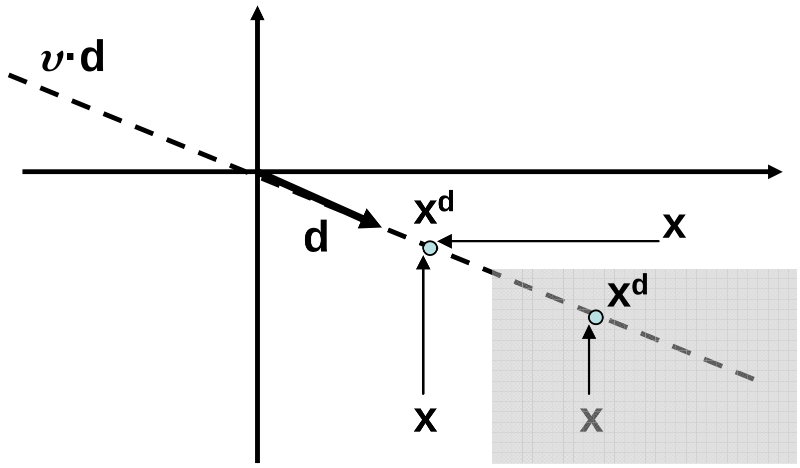


Figure 3: $MVaR_{\alpha}^{d}$ as an intersection of unidimensional VaRs.

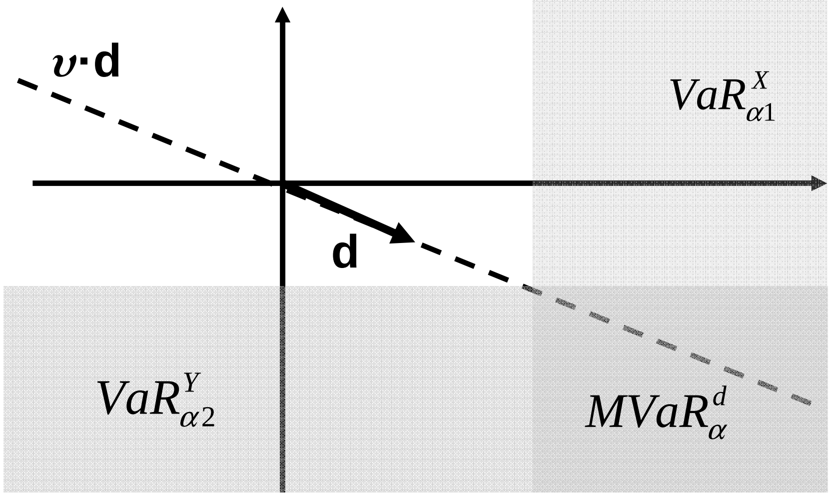
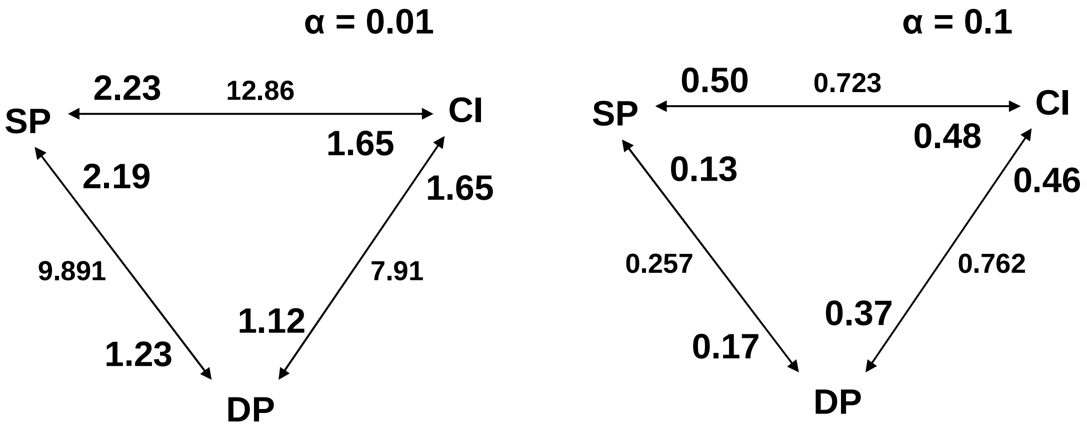


Figure 4: Conditional MVaRs and dependence coefficients.



Notes: Conditional MVaRs and dependence coefficients for the risk factors S&P 500 stock index (SP), CRB spot index (CI) and USD/GBP exchange rate (DP) computed for vectors $-(\sigma_{SP},0,0)$, $-(0,\sigma_{CI},0)$, $-(0,0,\sigma_{DP})$ at 1% and 10% significance level. The number in the middle of each double-arrow line is the dependence coefficient between the corresponding risk factors. The number close to a risk factor on a double-arrow line is this factors conditional MVaR, given the occurrence of the α-VaR for the risk factor on the opposite side of the same line.